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WEAK AND STRONG CONNECTIVITY REGIMES FOR A GENERAL TIME ELAPSED NEURON NETWORK MODEL

S. MISCHLER, C. QUIÑINAO, Q. WENG

ABSTRACT. For large fully connected neuron networks, we study the dynamics of homogenous assemblies of interacting neurons described by time elapsed models.

Under general assumptions on the firing rate which include the ones made in previous works [14, 15, 13], we establish accurate estimate on the long time behavior of the solutions in the weak and the strong connectivity regime both in the case with and without delay. Our results improve [14, 15] where a less accurate estimate was established and [13] where only smooth firing rates were considered.

Our approach combines several arguments introduced in the above previous works as well as a slightly refined version of the Weyl's and spectral mapping theorems presented in [21, 11].

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CONTENTS

1. Introduction	1
2. Existence of solutions	5
3. A Weyl's and spectral mapping theorem	10
4. Case without delay	12
5. Case with delay	17
References	19

Keywords. Neuron networks, time elapsed dynamics, semigroup, spectral analysis, weak connectivity, strong connectivity, exponential asymptotic stability.

1. INTRODUCTION

The information transmission and processing mechanism in the nervous systems relies on the quantity of electrical pulses as the reflect to incoming stimulations, during which the neurons experience a period of recalcitrance called discharge time before reactive (for more information about neuronal networks and mean-field approach see e.g. [19, 1]). In this work, we shall focus on the model describing the neuronal dynamics in accordance with this kind of discharge time which has been introduced and studied in [9, 14, 15]. In order to show the response to the recovery of the neuronal membranes after each discharge, the model consider an instantaneous firing rate depending on the time elapsed since last discharge as well as the inputs of neurons. This sort of models are also regarded as a mean field limit of finite number of neuron network models referred to [5, 8, 17, 6, 3, 10, 16].

For a local time (or internal clock) $x \geq 0$ corresponding to the elapsed time since the last discharge, we consider the dynamic of the neuronal network with the density number of neurons $f = f(t, x) \geq 0$ in state $x \geq 0$ at time $t \geq 0$, given by the following nonlinear time elapsed (or of age structured type) evolution equation

$$(1.1a) \quad \partial_t f = -\partial_x f - k(x, \lambda m(t))f =: \mathcal{L}_{\lambda m(t)} f,$$

$$(1.1b) \quad f(t, 0) = p(t), \quad f(0, x) = f_0(x),$$

where $k(x, \lambda \mu) \geq 0$ denotes the firing rate of a neuron in state x and in an environment $\mu \geq 0$ formed by the global neuronal activity with a network connectivity parameter $\lambda \geq 0$ corresponding to the strength of the interactions. The total density of neurons $p(t)$ undergoing a discharge at time t is defined through

$$(1.2a) \quad p(t) := \mathcal{P}[f(t); m(t)],$$

where

$$(1.2b) \quad \mathcal{P}[g, \mu] = \mathcal{P}_\lambda[g, \mu] := \int_0^\infty k(x, \lambda \mu) g(x) dx,$$

while the global neuronal activity $m(t)$ at time $t \geq 0$ taking into account the interactions among the neurons resulting from earlier discharges is given by

$$(1.3) \quad m(t) := \int_0^\infty p(t-y)b(dy).$$

Here the delay distribution b is a probability measure taking into account the persistence of the electric activity to those discharges in the network. In the sequel, we will consider the two following situations respectively:

- The *case without delay* when $b = \delta_0$, and then $m(t) = p(t)$;
- The *case with delay* when b is a smooth function.

Observe that the solution f of the time elapsed equation (1.1) satisfies

$$\frac{d}{dt} \int_0^\infty f(t, x) dx = f(t, 0) - \int_0^\infty k(x, \lambda m(t)) f(t, x) dx = 0,$$

in both cases. That implies the conservation of the total density number of neurons which can be thus normalized to 1. As a consequence, we assume in the sequel

$$(1.4) \quad \langle f(t, \cdot) \rangle = \langle f_0 \rangle = 1, \quad \forall t \geq 0, \quad \langle g \rangle := \int_0^\infty g(x) dx.$$

We call steady state a couple (F_λ, M_λ) of a nonnegative function and a positive real number which satisfies

$$(1.5a) \quad 0 = -\partial_x F_\lambda - k(x, \lambda M_\lambda) F_\lambda = \mathcal{L}_{\lambda M_\lambda} F_\lambda,$$

$$(1.5b) \quad F_\lambda(0) = M_\lambda, \quad \langle F_\lambda \rangle = 1.$$

Noticing that the associated network activity and the discharge activity are equal constants for a steady state because $\langle b \rangle = 1$.

Our main purpose in this paper is to prove existence, uniqueness and exponential asymptotic stability of solutions to the time elapsed evolution equation (1.1) in weak and strong connectivity regimes, which is a range of connectivity parameter $\lambda \in (0, \lambda_0) \cup (\lambda_\infty, \infty)$, with $\lambda_0 > 0$ small enough and $\lambda_\infty > 0$ large enough, chosen in such a way that the nonlinear term in equations (1.1) and (1.5) is not too strong.

These results are obtained for a rather large class of firing rate. More precisely, we make the physically reasonable assumptions

$$(1.6) \quad k \geq 0, \quad \partial_x k \geq 0, \quad k' = \partial_\mu k \geq 0,$$

meaning that neurons increases their firing rate as the global activity is higher and as the elapsed time since last decharge increases,

$$(1.7) \quad 0 < k_0 := \lim_{x \rightarrow \infty} k(x, 0) \leq \lim_{x, \mu \rightarrow \infty} k(x, \mu) =: k_1 < \infty,$$

as well as the regularity assumption

$$(1.8) \quad K(x, \cdot) := \int_0^x k(y, \cdot) dy \in C^0(\mathbb{R}_+), \quad \forall x > 0.$$

We will also need the stronger regularity assumption $k \in \text{Lip}_\mu L_x^1$. For the weak connectivity regime, we assume that for some $\xi > 0$ small enough and for any $\mu_0 > 0$, there exists $\lambda_0 > 0$ small enough such that

$$(1.9) \quad \int_0^\infty |k(x, \lambda \mu_2) - k(x, \lambda \mu_1)| dx \leq \xi |\mu_2 - \mu_1|, \quad \forall \mu_1, \mu_2 \in (0, \mu_0), \quad \forall \lambda \in (0, \lambda_0).$$

While in the strong connectivity regime, we assume that for some the same $\xi > 0$ as in the assumption (1.9) and for any $\mu_\infty > 0$, there exists $\lambda_\infty > 0$ large enough such that

$$(1.10) \quad \int_0^\infty |k(x, \lambda \mu_2) - k(x, \lambda \mu_1)| dx \leq \xi |\mu_2 - \mu_1|, \quad \forall \mu_1, \mu_2 \in (\mu_\infty, \infty), \quad \lambda \in (\lambda_\infty, \infty).$$

A possible example of firing rate which fulfills the above condition (1.9) for the weak connectivity regime is the ‘‘step function firing rate’’ considered in [14, 15] which is given by

$$(1.11) \quad k(x, \mu) = \mathbf{1}_{x > \sigma(\mu)}, \quad \sigma' \leq 0,$$

$$(1.12) \quad \sigma_+ := \sigma(0), \quad \sigma_- := \sigma(\infty), \quad \sigma_- < \sigma_+ < 1,$$

where σ satisfies the regularity condition

$$(1.13) \quad \sigma, \sigma^{-1} \in W^{1, \infty}(\mathbb{R}_+).$$

Similarly, the above condition (1.10) for the strong connectivity regime is met for a ‘‘step function firing rate’’ introduced in [14, 15] given by the same function as above which additionally fullfils

$$s |\sigma'(s)| \rightarrow 0 \quad \text{as } s \rightarrow \infty.$$

In the case with delay, we assume that the delay distribution is associated to a measurable function, namely $b(dy) = b(y)dy$ with $b \in L^1(\mathbb{R}_+)$, and satisfies the exponential bound

$$(1.14) \quad \exists \delta > 0, \quad \int_0^\infty e^{\delta y} b(y) dy < \infty.$$

Our first result establishes the existence and uniqueness of weak solution to the evolution problem (1.1). We call weak solution a function $0 \leq f \in C(\mathbb{R}_+; L^1(\mathbb{R}_+)_w) \cap L^\infty(\mathbb{R}_+^2)$ such that

$$\int_0^T \int_0^\infty f (\partial_t \varphi + \partial_x \varphi) dx dt = \int_0^T \int_0^\infty k(x, \lambda m) f \varphi dx dt - \int_0^T p \varphi(t, 0) dt$$

for any $\varphi \in C_c^1(\mathbb{R}_+^2)$, where p and m satisfy (1.2)-(1.3). Here and below $L_q^1(\mathbb{R}_+)$, for $q > 0$, stands for the space of L^1 functions f such that $x^q f \in L^1$ and L_w^1 denotes the $L^1(\mathbb{R}_+)$ space endowed with the weak topology $\sigma(L^1, L^\infty)$.

Theorem 1.1. *We consider a firing rate k satisfying (1.6)-(1.7)-(1.8) and a initial datum $0 \leq f_0 \in L^1_q(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$, for some $q > 0$, with total density number of neuron 1. We further assume that one of the following conditions holds:*

- (1) *the delay distribution b satisfies (1.14);*
- (2) *$b = \delta_0$, k satisfies (1.9) and $\lambda \in (0, \lambda_0)$, for $\lambda_0 > 0$ small enough;*
- (3) *$b = \delta_0$, k satisfies (1.10) and $\lambda \in (\lambda_\infty, \infty)$, for $\lambda_\infty > 0$ large enough, as well as*

$$(1.15) \quad \kappa_0 := \int_0^\infty k(x, 0) f_0(x) dx > 0.$$

In any of these three cases, there exists a weak solution $0 \leq f \in C(\mathbb{R}_+; L^1(\mathbb{R}_+)_w) \cap L^\infty(\mathbb{R}_+; L^1_q(\mathbb{R}_+)) \cap L^\infty(\mathbb{R}_+^2)$ to the evolution equation (1.1)-(1.2)-(1.3) for some functions $m, p \in C(\mathbb{R}_+)$ which satisfies the total number density of neurons conservation (1.4) as well as

$$(1.16) \quad \|f_t\|_{L^\infty} \leq \|f_0\|_{L^\infty} + k_1, \quad \|f\|_{L^1_q} \leq \|f_0\|_{L^1_q} + K_q, \quad \forall t \geq 0,$$

for some constant $K_q = K_q(k) \geq 0$ and

$$(1.17) \quad 0 \leq \kappa_1 \leq \|m_t\|_{L^\infty} \leq k_1, \quad \forall t \geq \tau,$$

with $\kappa_1 > 0$ when $\kappa_0 > 0$ or when $\tau > 0$ is large enough. The solution is furthermore unique in case (2) and (3).

Our proof is based on a Schauder fixed point theorem in the case with delay (1) and on a Banach fixed point theorem in cases without delay (2) and (3).

As a second step, we state an existence of solution to the stationary problem (1.5) and the uniqueness of that solution in the weak and strong connectivity regime.

Theorem 1.2. *Under the above assumption (1.6)-(1.7)-(1.8) on the firing rate, for any $\lambda \geq 0$, there exists at least one couple $(F_\lambda(x), M_\lambda) \in W^{1,\infty}(\mathbb{R}_+) \times \mathbb{R}_+$ solution to the stationary problem (1.5), and such that*

$$(1.18) \quad 0 \leq F_\lambda(x) \lesssim e^{-\frac{k_0}{2}x}, \quad |F'_\lambda(x)| \lesssim e^{-\frac{k_0}{2}x}, \quad x \geq 0.$$

Moreover, when we assume additionally that (1.9) and (1.10) hold, there exist $\lambda_0 > 0$ small enough and λ_∞ large enough, such that the above steady state is unique for any $\lambda \in [0, \lambda_0) \cup (\lambda_\infty, \infty]$.

The proof being identical to the ones presented in [14] and [13, Theorem 2.1], it will be skipped.

Finally our third and main result in the present paper states the exponential nonlinear stability of the above stationary state in the weak and strong connectivity regime.

Theorem 1.3. *We assume that k, b and f_0 satisfy the same conditions (1), (2) or (3) as in Theorem 1.1 and furthermore k satisfies (1.9) and (1.10). There exist $\lambda_0 > 0$ small enough, $\lambda_\infty > 0$ large enough, some constants $\alpha < 0$ and $C \geq 1$ such that for any $\lambda \in (0, \lambda_0) \cup (\lambda_\infty, +\infty)$ the solution f to the evolution equation (1.1)-(1.2)-(1.3) built in Theorem 1.1 furthermore satisfies*

$$(1.19) \quad \|f(t, \cdot) - F_\lambda\|_{L^1} \leq Ce^{\alpha t}, \quad \forall t \geq 0.$$

This theorem generalizes to the delay case the similar results obtained in [14, 15] and it generalizes the similar result obtained in [13] to a more general firing rate including the step function rate considered in [14, 15].

The proof is mainly based on an extension of the abstract semigroup theory developed in [11, 13] which has probably its own interest. It uses an auxiliary linear problem introduced in [14, 15] instead of the linearized equation considered in [13]. Both arguments together make possible to get ride of the smoothness assumption needed in [13] and moreover, allow us to consider the large connectivity regime, and also generalize the stability results established in [14, 15].

Our approach is thus quite different from the usual way to deal with delay equations which consists in using the specific framework of “fading memory space”, which goes back at least to Coleman & Mizel [4], or the theory of “abstract algebraic-delay differential systems” developed by O. Diekmann and co-authors [7]. It is also different from the previous works [14, 15] where the asymptotic stability analysis were performed by taking advantage of the “step function firing rate” (1.11), making possible to find a suitable norm such that the problem becomes dissipative.

This paper is organized as follows. In Section 2, we establish the existence and uniqueness results for the evolution equation as stated in Theorem 1.1. The estimate on the long time behavior of solutions as formulated in Theorem 1.3 is established in Section 4 in the case without delay. The case with delay is tackled in Section 5.

2. EXISTENCE OF SOLUTIONS

2.1. Delay case. In order to establish the existence of a solution to (1.1)-(1.2)-(1.3), we will apply a Schauder fixed point argument. To begin with, we analyze the continuity property of the functional \mathcal{P} defined in (1.2b).

Lemma 2.1. *Assume (1.6)-(1.7)-(1.8). Consider a sequence (m_n) of nonnegative real numbers converging to a limit m in \mathbb{R} as well as a sequence of functions (f_n) which converges to f in the sense of the weak topology $\sigma(L^1, L^\infty)$ and is uniformly bounded in L^∞ . We then have*

$$\mathcal{P}[f_n, m_n] \rightarrow \mathcal{P}[f, m], \quad \text{as } n \rightarrow \infty.$$

Proof. Step 1. Continuity of k . We are going to show that

$$(2.1) \quad k(\cdot, m_n) \rightarrow k(\cdot, m), \quad \text{a.e. as } n \rightarrow \infty.$$

We first assume that (m_n) is increasing. The sequence $(k(\cdot, m_n))$ is also increasing because of assumption (1.6). Moreover, since k is bounded from assumption (1.7), there exists some $\bar{k}(x)$ such that

$$k(x, m_n) \rightarrow \bar{k}(x), \quad \text{as } n \rightarrow \infty, \quad \text{for any } x \geq 0,$$

which in turn implies

$$K(x, m_n) = \int_0^x k(y, m_n) dy \rightarrow \int_0^x \bar{k}(y) dy.$$

From assumption (1.8), we deduce

$$\int_0^x k(y, m) dy = K(x, m) = \int_0^x \bar{k}(y) dy, \quad \forall x \geq 0.$$

Thus, we clearly have

$$\bar{k}(x) = k(x, m), \quad \text{for a.e. } x \geq 0.$$

The same holds in the case when (m_n) is a decreasing sequence. In the general case, we may define two monotonous sequences (m_n^i) , $i = 1, 2$, such that $m_n^1 \leq m_n \leq m_n^2$ for any $n \geq 1$ and such that $m_n^i \rightarrow m$ as $n \rightarrow \infty$ for $i = 1, 2$. Then $k(x, m_n^1) \leq k(x, m_n) \leq k(x, m_n^2)$

for any $n \geq 1$, $x \geq 0$ and $k(x, m_n^i) \rightarrow k(x, m)$ as $k \rightarrow \infty$ for a.e. $x \geq 0$ and for $i = 1, 2$. We immediately conclude that (2.1) holds.

Step 2. Continuity of the functional \mathcal{P} . We compute

$$\begin{aligned} \mathcal{P}[f_n, m_n] - \mathcal{P}[f, m] &= \int_0^\infty k(\cdot, m_n) f_n - \int_0^\infty k(\cdot, m) f \\ &= \int_0^\infty (k(\cdot, m_n) - k(\cdot, m)) f_n + \int_0^\infty k(\cdot, m) (f_n - f) \\ &:= I_1 + I_2. \end{aligned}$$

From the assumption (1.6) and the weak convergence of f_n in L^1 , we have $I_2 \rightarrow 0$, as $n \rightarrow \infty$. We write

$$I_1 = \int_0^R (k(x, m_n) - k(x, m)) f_n(x) dx + \int_R^\infty (k(x, m_n) - k(x, m)) f_n(x) dx.$$

From Step 1 and the assumption that (f_n) is bounded in L^∞ and uniformly integrable at the infinity (as a consequence of its weak $\sigma(L^1, L^\infty)$ convergence and the Dunford-Pettis theorem), we deduce that

$$|I_1| \leq \|f_n\|_{L^\infty} \int_0^R |k(x, m_n) - k(x, m)| dx + 2k_1 \int_R^\infty f_n(x) dx \rightarrow 0,$$

as $R \rightarrow \infty$ and $n \rightarrow \infty$. The two above estimates together imply the conclusion. \square

In a next step, we fix $T > 0$ and we analyse the linear mapping which associates to a given function $m \in C([0, T])$ the solution $f \in C([0, T]; L^1) \cap L^\infty([0, T]; L^\infty)$ to the transport equation

$$(2.2) \quad \begin{aligned} \partial_t f + \partial_x f + k(x, m(t)) f &= 0 \\ f(t, 0) = \mathcal{P}[f, m(t)], \quad f(0, x) &= f_0(x). \end{aligned}$$

The following lemma gives the continuity of this mapping.

Lemma 2.2. *Fix $T > 0$. Consider a sequence (m_n) such that $m_n \rightarrow \bar{m}$ in $C([0, T])$, as $n \rightarrow \infty$. There exists then a sequence (f_n) of solutions to the linear transport equation (2.2) associated to (m_n) and this one satisfies*

$$f_n \rightarrow \bar{f} \quad \text{in } C([0, T]; L_w^1) \cap L^\infty([0, T]; L^\infty \cap L_q^1), \quad \text{as } n \rightarrow \infty,$$

where \bar{f} stands for the solution to the linear transport equation (2.2) associated to \bar{m} .

Proof. Step 1. Existence of f_n . For any $m \in C([0, T])$ and any $0 \leq g_1 \in X_T := C([0, T]; L^1) \cap L^\infty(0, T; L^\infty \cap L_q^1)$ we may associate $0 \leq g_2 \in X_T$ as the solution to the equation

$$(2.3) \quad \begin{aligned} \partial_t g_2 + \partial_x g_2 + k(x, m(t)) g_2 &= 0 \\ g_2(t, 0) = p_1(t) := \mathcal{P}[g_1, m(t)], \quad g_2(0, x) &= f_0(x), \end{aligned}$$

which is classically defined through the characteristic method. More precisely, we introduce the space

$$\mathcal{C} := \{0 \leq g \in X_T; \|g\|_{L^1} \leq e^{k_1 t}, \|g\|_{L^\infty} \leq \|f_0\|_{L^\infty} + k_1, \|g\|_{L_q^1} \leq C_q \|f_0\|_{L_q^1} + e^{k_1 T}\},$$

for some constant $C_q \geq 1$ that we specify below and we consider $g_1 \in \mathcal{C}$. Integrating the equation (2.3) on x , we find

$$\frac{d}{dt} \int_0^\infty g_2(t, x) dx = \int_0^\infty (g_1(t, x) - g_2(t, x)) k(x, m(t)) dx \leq k_1 \|g_1\|_{L^1} \leq k_1 e^{k_1 t},$$

which implies

$$\|g_2\|_{L^1} \leq \int_0^t k_1 e^{k_1 s} ds + \|f_0\|_{L^1} = e^{k_1 t}.$$

We define

$$\omega(x) := \mathbf{1}_{x \leq x_0} + \frac{x^q}{x_0^q} \mathbf{1}_{x > x_0},$$

with $x_0 \geq 1$ large enough such that $q/x - k(\mu, x) \leq q/x_0 - k(0, x_0) \leq -k_0/2 \leq 0$ for any $x \geq x_0$ and $\mu \geq 0$. Similarly as above, we deduce

$$\begin{aligned} \frac{d}{dt} \int_0^\infty g_2 \omega &= \int_0^\infty k(x, m(t)) g_1 + \int_0^\infty g_2 [\partial_x \omega - k(x, m(t)) \omega] \\ &\leq k_1 e^{k_1 t}, \end{aligned}$$

which implies

$$\|g_2\|_{L_\omega^1} \leq e^{k_1 t} - 1 + \|f_0\|_{L_\omega^1} \leq e^{k_1 t} + x_0^{-q} \|f_0\|_{L_q^1}.$$

We finally get

$$\|g_2\|_{L_q^1} \leq C_q e^{k_1 t} + \|f_0\|_{L_q^1},$$

with $C_q := x_0^q$. Thanks to the method of characteristics the solution $g_2(t, x)$ to equation (2.3) can be expressed as

$$g_2(t, x) = \begin{cases} f_0(x-t) e^{-\int_0^t k(s+x-t, m(s)) ds}, & \forall x \geq t, \\ p_1(t-x) e^{-\int_0^x k(s, m(s+t-x)) ds}, & \forall x \leq t, \end{cases}$$

which implies

$$\|g_2\|_{L^\infty} \leq \|f_0\|_{L^\infty} + \|p_1\|_{L^\infty} \leq \|f_0\|_{L^\infty} + k_1.$$

Denoting $g_2 := \mathcal{I}(g_1)$, we have proved $\mathcal{I}(\mathcal{C}) \subset \mathcal{C}$. On the other hand, denoting $h_1 = g_1 - \tilde{g}_1$ and $h_2 := \mathcal{I}(g_1) - \mathcal{I}(\tilde{g}_1)$ for $g_1, \tilde{g}_1 \in \mathcal{C}$, a similar computation as above leads to

$$\sup_{t \in [0, T]} \|h_2(t)\|_{L^1} \leq \frac{1}{k_1} (e^{k_1 T} - 1) \sup_{t \in [0, T]} \|h_1(t)\|_{L^1},$$

from what we conclude to the existence of a unique function $f \in \mathcal{C}$ such that $\mathcal{I}(f) = f$ by a classical contraction fixed point Theorem for $T > 0$ small enough. We get $T > 0$ arbitrary by iterating the argument. We thus get the existence of the sequence (f_n) and the possible limit \bar{f} by applying the above construction with $m = m_n$ and $m = \bar{m}$.

Step 2. Uniform estimates on \bar{f} . By integrating the transport equation (2.2), we obtain that the solution conserves the total density number of neurons as given by (1.4). For any solution \bar{f} to the equation (2.2), we deduce

$$\begin{aligned} \frac{d}{dt} \int_0^\infty \bar{f} \omega &= \int_0^\infty k(x, m(t)) \bar{f} + \int_0^\infty \bar{f} [\partial_x \omega - k(x, m(t)) \omega] \\ &\leq k_1 - \frac{k_0}{2} \int_{x_0}^\infty \bar{f} \omega \leq k_1 + \frac{k_0}{2} - \frac{k_0}{2} \int_0^\infty \bar{f} \omega, \end{aligned}$$

which implies

$$\|\bar{f}\|_{L_\omega^1} \leq \min\left\{1 + \frac{2k_1}{k_0}, \|f_0\|_{L_\omega^1}\right\}.$$

We finally define

$$\bar{\mathcal{C}} := \{0 \leq f \in X_T; \|f\|_{L^1} = 1, \|f\|_{L^\infty} \leq \|f_0\|_{L^\infty} + k_1, \|f\|_{L_q^1} \leq \|f_0\|_{L_q^1} + K_q\},$$

with $K_q := 2x_0^q(1 + k_1/k_0)$. By construction, \bar{f} obviously belongs to $\bar{\mathcal{C}}$.

Step 3. Continuity of the mapping. From equation (2.2), we observe that

$$\partial_t f_n \leq -\partial_x f_n, \quad f_n(t, 0) \leq k_1, \quad f_n(0, x) = f_0,$$

which implies

$$f_n(t, x) \leq k_1 \mathbf{1}_{x \leq t} + f_0(x - t) \mathbf{1}_{x \geq t},$$

for any $t \leq T$. From this upper bound, one gets

$$\int_R^\infty f_n \leq \int_R^\infty f_0(x - t) dx,$$

for any $R \geq T$, and in particular

$$\int_R^\infty f_n \leq \int_{R-T}^\infty f_0 \rightarrow 0,$$

as $R \rightarrow \infty$ and uniformly in $n \geq 1$. From equation (2.3), for any $\varphi \in C_c^1((0, \infty))$, we also have

$$\frac{d}{dt} \int_0^\infty f_n \varphi dx = A_n^\varphi,$$

with A_n^φ bounded in $L^\infty(0, T)$ uniformly in $n \geq 1$. Together with the fact that $f_n \in \mathcal{C}$ for any $n \geq 1$, we may use the de la Vallée-Poussin Theorem and the Dunford-Pettis Lemma to conclude that there exists $f \in C([0, T]; L_w^1) \cap L^\infty([0, T]; L^\infty)$ and a subsequence $f_{n'}$ of the sequence f_n such that $f_{n'} \rightharpoonup f$ weakly. We deduce $\mathcal{P}[f_{n'}, m_{n'}] \rightarrow \mathcal{P}[f, m]$ as $n' \rightarrow \infty$, from Lemma 2.1. We finally conclude by passing n' to the limit in the equation (2.2) with $m_{n'}$. \square

Proof of Theorem 1.1 - the delay case. We recall that $b \in L^1(\mathbb{R}_+)$ in that case. We consider the application $\mathcal{J} : C([0, T]) \rightarrow C([0, T])$, defined as

$$\mathcal{J}(m)(t) := \int_0^t p(t - y) b(dy), \quad \forall m \in C([0, T]), \quad \forall t \in [0, T],$$

where $p(t) = \mathcal{P}[f(t, \cdot), m(t)]$ and $f \in \bar{\mathcal{C}}$ is a solution to (2.2) which existence has been established during the proof of Lemma 2.2. From Lemma 2.1, we deduce that the application $m \rightarrow \mathcal{P}[f, m]$ is continuous and so is \mathcal{J} . Define

$$\mathcal{K} := \{m \in C([0, T]), \|m\|_{L^\infty} \leq k_1\}.$$

Obviously, \mathcal{K} is a convex subset of $C([0, T])$ and, for any $m \in \mathcal{K}$, we have

$$\|\mathcal{J}(m)\|_{L^\infty} \leq \int_0^t |b(y)| \|p\|_{L^\infty} dy \leq k_1,$$

so that $\mathcal{J} : \mathcal{K} \rightarrow \mathcal{K}$. On the other hand, for any $\epsilon > 0$, there exists $\theta > 0$ such that for any $t, s \in [0, T]$ satisfying $|t - s| < \theta$, we have

$$\begin{aligned} |\mathcal{J}(m)(t) - \mathcal{J}(m)(s)| &\leq \int_0^\infty |b(t - y) - b(s - y)| \|p\|_{L^\infty} dy \\ &\leq \|b(t - \cdot) - b(s - \cdot)\|_{L^1(\mathbb{R}_+)} k_1 \|f_0\|_{L^1} \\ &\leq k_1 \|\tau_\theta b - b\|_{L^1(\mathbb{R}_+)} < \epsilon, \end{aligned}$$

where $\tau_\theta b := b(\cdot + \theta)$, which implies that \mathcal{J} is equicontinuous. Thanks to the Arzela-Ascoli Theorem, we deduce that $\mathcal{J}(\mathcal{K})$ is compactly embedded into \mathcal{K} . Using the Schauder-Brouwer fixed point Theorem, the application \mathcal{J} admits a fixed point $m \in \mathcal{K}$. The corresponding solution f to the equation (2.2) is also a solution to the equation (1.1)-(1.2)-(1.3). Iterating on T , we deduce the existence of a global solution (f, m, p) to equation (1.1)-(1.3), which satisfies the bound in Theorem 1.1.

In order to prove the lower bound (1.17) on m , we recall that $x_0 \geq 1$ has been defined such that

$$k(x, \mu) \geq \frac{k_0}{2} \mathbf{1}_{x \geq x_0}, \quad \forall x > 0, \mu > 0.$$

For $t \leq x_0$, thanks to the characteristics expression, we have

$$\begin{aligned} p(t) &\geq \int_0^\infty k(x, 0) f_0(x-t) e^{-\int_0^t k(s+x-t, m(s)) ds} dx \\ &\geq \int_0^\infty k(x, 0) f_0(x) e^{-k_1 t} dx \\ &\geq e^{-k_1 x_0} \kappa_0. \end{aligned}$$

We consider now the case $t > x_0$. Directly from (1.1), we have

$$\partial_t f \geq \partial_x f - k_1 f,$$

which implies

$$f(t, x) \geq e^{-k_1 x_0} f(t - x_0, x - x_0).$$

We then deduce

$$\begin{aligned} p(t) &\geq e^{-k_1 x_0} \int_0^\infty \frac{k_0}{2} \mathbf{1}_{x \geq x_0} f(t - x_0, x - x_0) dx \\ &\geq \frac{k_0}{2} e^{-k_1 x_0} \int_0^\infty f(t - x_0, x) dx \\ &= \frac{k_0}{2} e^{-k_1 x_0}. \end{aligned}$$

All together, we deduce the same lower bound for $m(t)$ from the definition (1.3). \square

2.2. Without delay case. We will need the following auxilliary result. We define the function $\Phi : L^1(\mathbb{R}_+) \times \mathbb{R} \rightarrow \mathbb{R}$ by

$$\Phi[g, \mu] := \int_0^\infty k(x, \lambda \mu) g(x) dx - \mu.$$

Lemma 2.3. *Assume (1.6)-(1.7)-(1.8)-(1.9)-(1.10). For any bounded set $\mathcal{C} \subset \mathbf{P} \cap L^\infty$, there exists $\lambda_0 = \lambda_0(\mathcal{C}) > 0$ and $\lambda_\infty = \lambda_\infty(\mathcal{C}) > 0$, such that for any $\mu_0 > 0$ and $\mu_\infty > 0$, there exists a function $\varphi_\lambda : \mathcal{C} \rightarrow \mathbb{R}$ which is Lipschitz continuous in the sense of the L^1 norm and such that $\mu = \varphi_\lambda[g]$ is the unique solution to the equation*

$$\begin{aligned} \mu \in (0, \mu_0), \quad \Phi(g, \mu) &= 0, \quad \forall \lambda \in (0, \lambda_0), \\ \mu \in (\mu_\infty, k_1), \quad \Phi(g, \mu) &= 0, \quad \forall \lambda \in (\lambda_\infty, \infty). \end{aligned}$$

Proof of Lemma 2.3. The proof is similar to the one of [13, Lemma 2.8], thus we skip the existence part and we present the uniqueness part here. Fix $R > 0$ and take $f, g \in L^1 \cap L^\infty$ and $\mu, \nu \in (0, \mu_0)$ or $\mu, \nu \in (\mu_\infty, k_1)$ such that

$$\|f\|_{L^\infty} \leq R \quad \text{and} \quad \Phi(f, \mu) = \Phi(g, \nu) = 0.$$

We have

$$\nu - \mu = \int_0^\infty k(x, \lambda \nu) (g - f) dx + \int_0^\infty (k(x, \lambda \nu) - k(x, \lambda \mu)) f dx,$$

with

$$\left| \int_0^\infty k(x, \lambda \nu) (g - f) dx \right| \leq k_1 \|f - g\|_{L^1}.$$

From the assumption (1.9)-(1.10) and the uniform estimate on f , there holds

$$\left| \int_0^\infty (k(x, \lambda \nu) - k(x, \lambda \mu)) f dx \right| \leq R \xi |\mu - \nu|, \quad \forall \lambda \in [0, \lambda_0) \cup (\lambda_\infty, \infty].$$

We then deduce

$$|\mu - \nu| \leq 2k_1 \|f - g\|_{L^1},$$

for any $\lambda \in [0, \lambda_0) \cup (\lambda_\infty, \infty]$, with $\lambda_0 = \lambda_0(R) > 0$ small enough and $\lambda_\infty = \lambda_\infty(R) > 0$ large enough. That implies the uniqueness of the solution $\mu = \varphi_\lambda(f) \in \mathbb{R}$ to the constraint problem $\Phi(f, \mu) = 0$ for any given $f \in L^1 \cap L^\infty$ and the Lipschitz continuity of φ_λ .

Proof of Theorem 1.1 - The case without delay. We fix $\lambda_0, \lambda_\infty > 0$ as defined in Lemma 2.3. For a given function $m \in C([0, T])$, we define $M(t) := \varphi_\lambda(f(t, \cdot)) \in C([0, T])$, where $f \in C([0, T]; L^1(\mathbb{R}_+)) \cap L^\infty([0, T] \times \mathbb{R}_+)$ is the solution of (2.2) associated to m . We denote $\mathcal{I}(m) := M$. For two given $m_1, m_2 \in C([0, T])$, we denote $f_1, f_2 \in C([0, T]; L^1(\mathbb{R}_+)) \cap L^\infty([0, T] \times \mathbb{R}_+)$ the associated solutions to (2.2) and we easily compute

$$\begin{aligned} \frac{d}{dt} \int |f_2 - f_1| &\leq 2 \int |k(m_2)f_2 - k(m_1)f_1| \\ &\leq 2\|f_0\|_{L^\infty} \xi |m_2 - m_1| + 2k_1 \int |f_2 - f_1|, \quad \forall \lambda \in (0, \lambda_0) \cup (\lambda_\infty, \infty). \end{aligned}$$

We deduce that $m \mapsto f$ is Lipschitz from $C([0, T])$ to $C([0, T]; L^1(\mathbb{R}_+))$ with constant CT . As a consequence, \mathcal{I} is Lipschitz from $C([0, T])$ into itself with constant $C'T$. Choosing $T > 0$ small enough, the mapping \mathcal{I} is a contraction and admits a unique fixed point thanks to the Banach fixed point theorem. Iterating on T , we deduce the existence and uniqueness of a global solution (f, m) to equation (1.1)-(1.2) in the case without delay in both weak and strong connectivity regimes. \square

3. A WEYL'S AND SPECTRAL MAPPING THEOREM

In this section we establish a simple version of Weyl's and spectral mapping theorem for semigroup in an abstract setting which slightly generalizes the versions of the same theorems established in [11, 13]. More precisely, we consider the generators \mathcal{L} and \mathcal{B} of two semigroups $S_{\mathcal{L}}$ and $S_{\mathcal{B}}$ in a Banach space \mathcal{X} . We denote $\mathcal{A} := \mathcal{L} - \mathcal{B}$ as well as $R_{\mathcal{L}}(z) := (\mathcal{L} - z)^{-1}$ $R_{\mathcal{B}}(z) := (\mathcal{B} - z)^{-1}$ the resolvent operators defined in the corresponding resolvent sets. We assume that for some fixed $a^* \in \mathbb{R}$ the following growth and regularizing estimates hold true for any $a > a^*$:

(H1) \mathcal{B} is \mathcal{A} -power dissipative in X , in the sense that

$$(3.1) \quad \forall \ell \geq 0, \quad t \mapsto \|S_{\mathcal{B}} * (\mathcal{A}S_{\mathcal{B}})^{(*\ell)}(t)\|_{\mathcal{B}(\mathcal{X})} e^{-at} \in L^\infty(0, \infty)$$

and $u := S_{\mathcal{B}}\mathcal{A}$ satisfies

$$(3.2) \quad \exists n \geq 1, C \in (0, \infty), \forall f \in \mathcal{X}, \quad \int_0^\infty \|u^{(*n)}(t)f\|_{\mathcal{X}} e^{-at} dt \leq C \|f\|_{\mathcal{X}}.$$

(H2) For the same integer $n \geq 1$, the operator $\mathcal{U} := -R_{\mathcal{B}}\mathcal{A}$ is power regular in the sense that

$$(3.3) \quad \exists \alpha > 0, C \in (0, \infty), \quad \|\mathcal{U}(z)^n\|_{\mathcal{B}(\mathcal{X})} \leq C \langle x \rangle^{-\alpha}, \quad \forall z \in \Delta_a$$

and

$$(3.4) \quad \forall M > 0, \exists C \in (0, \infty), \quad \|\mathcal{U}(z)^n\|_{\mathcal{B}(\mathcal{X}, \mathcal{Y})} \leq C, \quad \forall z \in \Delta_a \cap B(0, M),$$

for some linear space \mathcal{Y} such that the embedding $\mathcal{Y} \subset \mathcal{X}$ is compact.

For a given operator L we denote $\Sigma(L)$ its spectral set and we define $\Sigma_d(L)$ the discrete spectrum as the set of isolated eigenvalues with finite dimensional associated eigenspace. We also denote $D(L)$ the domain and RL the range.

Theorem 3.1. *We make the above growth and regularizing assumptions (H1) and (H2) on \mathcal{A} and \mathcal{B} for some $a^* < 0$ and we assume furthermore that*

$$\Sigma(\mathcal{L}) \cap \bar{\Delta}_0 = \{0\} \subset \Sigma_d(\mathcal{L}),$$

so that there exist a finite rank projector $\Pi_0 \in \mathcal{B}(\mathcal{X})$ and an operator $T_0 \in \mathcal{B}(R\Pi_0)$ satisfying $\mathcal{L}\Pi_0 = \Pi_0\mathcal{L} = T_0\Pi_0$, $\Sigma(T_0) = \{0\}$. There exist $a < 0$ and $C \geq 1$ such that

$$(3.5) \quad \forall t \geq 0, \quad \left\| e^{t\mathcal{L}} - e^{tT_0} \Pi_0 \right\|_{\mathcal{B}(\mathcal{X})} \leq C e^{at}.$$

For two given time dependent operators valued functions U and V , we define the convolution product

$$(U * V)(t) := \int_0^t U(t-s)V(s) ds.$$

We also denote $V^{(*1)} = V$ and $V^{(*\ell)} := V * V^{(*(\ell-1))}$ for any $\ell \geq 2$.

Proof of Theorem 3.1. Step 1. We define

$$\mathcal{V}(z) := \mathcal{R}_{\mathcal{B}}(z) - \dots + (-1)^{n-1} \mathcal{R}_{\mathcal{B}}(z) (\mathcal{A}\mathcal{R}_{\mathcal{B}}(z))^{n-1}$$

and

$$\mathcal{W}(z) := (-1)^n (\mathcal{R}_{\mathcal{B}}(z)\mathcal{A})^n,$$

where $n \geq 1$ is the integer given by assumption (H1). From the definition $\mathcal{L} = \mathcal{A} + \mathcal{B}$, we immediately have

$$R_{\mathcal{L}} = R_{\mathcal{B}} - R_{\mathcal{B}}\mathcal{A}R_{\mathcal{L}},$$

and by iterating that relation, we deduce

$$R_{\mathcal{L}} = \mathcal{V} + \mathcal{W}R_{\mathcal{L}},$$

or equivalently

$$(I - \mathcal{W})R_{\mathcal{L}} = \mathcal{V}.$$

Thanks to (3.3), for M large enough, we have

$$z \in \Delta_a, |z| \geq M \quad \Rightarrow \quad \|\mathcal{W}(z)\|_{\mathcal{B}(\mathcal{X})} \leq \frac{1}{2}.$$

We get that $R_{\mathcal{L}}(z) = (I - \mathcal{W}(z))^{-1}\mathcal{V}(z)$ is well defined and uniformly bounded in the region $\Delta_a \setminus B(0, M)$, or in other word $\Sigma(\mathcal{L}) \cap \Delta_a \subset B(0, M)$.

On the other hand, $\Phi := I - \mathcal{W}$ is holomorphic on Δ_{a^*} and $R(\mathcal{W}) \subset \mathcal{Y} \subset \mathcal{X}$ because of (3.1)-(3.4). Together with $\Phi(M)$ is invertible, we may use Ribarič-Vidav-Voigt theory [18, 21] and deduce that $R_{\mathcal{L}} = (I - \mathcal{W})^{-1}\mathcal{V}$ is a degenerate-meromorphic operator and next that $\Sigma(\mathcal{L}) \cap \Delta_{a^*}$ is discrete. All together, we have proved that there exists $a < 0$ such that $\bar{\Delta}_a \cap \Sigma(\mathcal{L}) = \{0\}$.

Step 2. For any integer $N \geq 1$ and iterating the Duhamel formula

$$S_{\mathcal{L}} = S_{\mathcal{B}} + (S_{\mathcal{B}}\mathcal{A}) * S_{\mathcal{L}},$$

we have

$$S_{\mathcal{L}}(I - \Pi_0) = \sum_{\ell=0}^{N-1} S_{\mathcal{B}} * (\mathcal{A}S_{\mathcal{B}})^{(*\ell)}(I - \Pi_0) + (S_{\mathcal{B}}\mathcal{A})^{(*N)} * (S_{\mathcal{L}}(I - \Pi_0)).$$

For $b > \Lambda(\mathcal{L})$, we may use the inverse Laplace formula

$$\begin{aligned} \mathcal{T}(t)f &:= (\mathcal{AS}_{\mathcal{B}})^{(*N)} * (S_{\mathcal{L}}(I - \Pi_0))(t)f \\ &= \lim_{M' \rightarrow \infty} \frac{i}{2\pi} \int_{b-iM'}^{b+iM'} e^{zt} (-1)^{N+1} (\mathcal{R}_{\mathcal{B}}(z)\mathcal{A})^N (I - \Pi_0)\mathcal{R}_{\mathcal{L}}(z) f dz, \end{aligned}$$

for any $f \in D(\mathcal{L})$ and $t \geq 0$, and we emphasize that the term $\mathcal{T}(t)f$ might be only defined as a semi-convergent integral. Because $z \mapsto (\mathcal{R}_{\mathcal{B}}(z)\mathcal{A})^N (I - \Pi_0)\mathcal{R}_{\mathcal{L}}(z)$ is a bounded analytic function on a neighborhood of $\bar{\Delta}_a$, we may move the segment on which the integral is performed, and we obtain

$$(3.6) \quad \mathcal{T}(t)f = \lim_{M' \rightarrow \infty} \frac{i}{2\Pi_0} \int_{a-iM'}^{a+iM'} e^{zt} (-1)^{N+1} (\mathcal{R}_{\mathcal{B}}(z)\mathcal{A})^N \mathcal{R}_{\mathcal{L}}(z) (I - \Pi_0) f dz,$$

for any $f \in D(\mathcal{L})$ and $t \geq 0$. In order to conclude we only have to explain why the RHS term in (3.6) is appropriately bounded for N large enough. We define

$$\mathcal{W}(z) := \mathcal{R}_{\mathcal{L}}(z) (\mathcal{AR}_{\mathcal{B}}(z))^N$$

for $z \in \Delta_a \setminus B(0, M)$, $N := ([1/\alpha] + 1)n$. From Step 1 and (3.3), we deduce

$$(3.7) \quad \|\mathcal{W}(z)\|_{\mathcal{B}(X)} \leq \frac{C}{|y|^\beta}, \quad \forall z = a + y, |y| \geq M,$$

with $\beta := ([1/\alpha] + 1)\alpha > 1$. We then have

$$\begin{aligned} \|\mathcal{T}(t)f\|_{\mathcal{B}(X)} &\leq \frac{e^{at}}{2\pi} \int_{a-iM}^{a+iM} \|(\mathcal{R}_{\mathcal{B}}(z)\mathcal{A})^N\|_{\mathcal{B}(X)} \|(I - \Pi_0)\mathcal{R}_{\mathcal{L}}(z)\|_{\mathcal{B}(X)} dy \\ &\quad + \frac{e^{at}}{2\pi} \int_{\mathbb{R} \setminus [-M, M]} \|\mathcal{W}(a + iy)(I - \Pi_0)\|_{\mathcal{B}(X)} dy, \end{aligned}$$

where the first integral is finite thanks to $\Sigma(\mathcal{L}(I - \Pi_0)) \cap [a - iM, a + iM] = \emptyset$ and (3.3), while the second integral is finite because of (3.7). \square

4. CASE WITHOUT DELAY

In this section, we present the proof of our main result Theorem 1.3 in the case without delay.

4.1. An auxiliary linear equation. We introduce the auxiliary linear equation on the variation g given by

$$(4.1) \quad \begin{aligned} \partial_t g + \partial_x g + k_\lambda g &= 0, \\ g(t, 0) &= \mathcal{M}_\lambda[g(t, \cdot)], \quad g(0, x) = g_0(x), \end{aligned}$$

with the notations

$$(4.2) \quad \mathcal{M}_\lambda[h] := \int_0^\infty k_\lambda h dx, \quad k_\lambda := k(x, \lambda M_\lambda),$$

and where M_λ is defined in Theorem 1.2. The corresponding linear operator \mathcal{L} is

$$\mathcal{L}g := -\partial_x g - k_\lambda g$$

in the domain

$$D(\mathcal{L}) := \{g \in W^{1,1}(\mathbb{R}_+), g(0) = \mathcal{M}_\lambda[g]\}$$

generating the semigroup $S_{\mathcal{L}}$ in the Lebesgue space $X := L^1(\nu)$ for some polynomial weight function $\nu := 1 + x^q$, $q > 0$. For any initial datum $g_0 \in X$, the weak solution of

the linearized equation is given by $g(t) = S_{\mathcal{L}}(t)g_0$. By regarding the boundary condition as a source term, we may rewrite the above equation as

$$(4.3) \quad \partial_t g = \Lambda g := -\partial_x g - k_\lambda g + \delta_{x=0} \mathcal{M}_\lambda[g],$$

with the associated semigroup S_{Λ_λ} , acting on the space of bounded Radon measures

$$\mathcal{X} := M^1(\mathbb{R}_+) = \{g \in (C_0(\mathbb{R}))'; \text{supp } g \subset \mathbb{R}_+\},$$

endowed with the weak $*$ topology $\sigma(M^1, C_0)$, where C_0 represents the space of continuous functions converging to 0 at infinity.

Theorem 4.1. *For any $\lambda \geq 0$, there exist $\alpha < 0$ and $C > 0$ such that $\Sigma(\mathcal{L}) \cap \Delta_\alpha = \{0\}$ and*

$$(4.4) \quad \|S_{\mathcal{L}}(t)g_0\|_X \leq C e^{\alpha t} \|g_0\|_X, \quad \forall t \geq 0,$$

for any $g_0 \in X$, $\langle g_0 \rangle = 0$.

We proceed in several steps.

Lemma 4.2. *The semigroup $S_{\mathcal{L}}$ is well defined in L^1 and it is positive in the sense that $S_{\mathcal{L}}(t)f_0 \geq 0$ for any $f_0 \in L^1$, $f_0 \geq 0$ and any $t \geq 0$.*

The proof being exactly the same as for [13, Lemma 2.5] it is skipped.

Lemma 4.3. *$-\mathcal{L}$ satisfies the following version of the strong maximum principle: for any given $g \in \mathcal{X}_+$ and $\mu \in \mathbb{R}$, there holds*

$$g \in D(\mathcal{L}) \setminus \{0\} \text{ and } (-\mathcal{L} + \mu)g \geq 0 \text{ imply } g > 0.$$

The proof being exactly the same as for [13, Lemma 2.6] it is skipped. As an immediate consequence of Theorem 1.2, Lemma 4.2 and Lemma 4.3, we obtain the following result about the first eigenvalue and eigenspace associated to \mathcal{L} . We refer to [13, Proof of Theorem 2.4.] or [11, 12, Proof of Theorem 5.3] where similar results are established.

Corollary 4.4. *There hold $\Sigma(\mathcal{L}) \cap \bar{\Delta}_0 = \{0\}$ and $N(\mathcal{L}) = \text{span}(F_\lambda)$.*

We come to the slightly new argument we need in order to generalize the proof presented in [13] to the non smooth firing rate we are considered here. Because of (1.6)-(1.7), we have

$$k_\lambda \in L^\infty(\mathbb{R}_+), \quad k_\lambda(x) \geq k_0/2 \mathbf{1}_{x \geq x_0},$$

for some $x_0 \in [0, \infty)$, and we set

$$a^* := -k_0/2 < 0.$$

We rewrite the evolution equation as

$$(4.5) \quad \partial_t f = \mathcal{L}f = \mathcal{A}f + \mathcal{B}f,$$

with \mathcal{A} and \mathcal{B} defined by

$$\begin{aligned} (\mathcal{A}f)(x) &:= \delta_{x=0} \mathcal{K}[f], \quad \mathcal{K}[f] := \int_0^\infty k_\lambda(y) f(y) dy \\ (\mathcal{B}f)(x) &:= -\partial_x f(x) - k_\lambda(x) f(x), \end{aligned}$$

and we emphasize that the boundary condition in (4.1) has been equivalently replaced by the term $\mathcal{A}f$ involving a Dirac mass $\delta_{x=0}$.

Lemma 4.5. For any $a > a^*$ and any $R > 0$, the operators \mathcal{A} and \mathcal{B} satisfy :

- (i) the operator $\mathcal{B} - a$ is dissipative in $L^1(\tilde{\nu})$, with norm equivalent to the norm of X ;
- (ii) the operators valued function of time $t \mapsto (S_{\mathcal{B}}\mathcal{A})^{(*\ell)} * S_{\mathcal{B}}(t) e^{-at}$ is bounded in $L^\infty(0, \infty; \mathcal{B}(X))$;
- (iii) the operators valued holomorphic function $z \mapsto \langle z \rangle (R_{\mathcal{B}}(z)\mathcal{A})^2(z)$ is bounded in $\mathcal{B}(X)$ uniformly in $z \in \Delta_a$;
- (iv) $z \mapsto (R_{\mathcal{B}}(z)\mathcal{A})^2(z)$ is bounded in $\mathcal{B}(X, Y)$ uniformly in $z \in \Delta_a \cap B(0, R)$, with $Y := BV \cap L^1_{q+1}$.

Proof of Lemma 4.5. During the proof we write $k = k_\lambda$.

Step 1. In order to prove the first point, we fix $a > a^*$ and we introduce the modified weight function

$$\tilde{\nu}(x) := \frac{e^{ax}}{e^{ax_1}} \mathbf{1}_{x \leq x_1} + \frac{x^q}{x_1^q} \mathbf{1}_{x > x_1},$$

with $x_1 > \max(1, q/(a - a^*))$. We compute

$$\begin{aligned} \mathcal{B}^* \tilde{\nu} &= \partial_x \tilde{\nu} - k \tilde{\nu} \leq a \tilde{\nu} \quad \text{on } [0, x_1] \\ \mathcal{B}^* \tilde{\nu} &\leq \left(\frac{q}{x} - k \right) \tilde{\nu} \leq a \tilde{\nu} \quad \text{on } (x_1, \infty), \end{aligned}$$

from what we deduce

$$\int_0^\infty (\mathcal{B}f) f / |f| \tilde{\nu} = \int_0^\infty (\mathcal{B}^* \tilde{\nu}) |f| \leq a \int_0^\infty |f| \tilde{\nu}$$

and $\mathcal{B} - a$ is dissipative in $L^1(\tilde{\nu})$.

Step 2. From the first step we have $\|S_{\mathcal{B}}(t)\|_{\mathcal{B}(X)} = \mathcal{O}(e^{at})$. We deduce (ii) recursively.

Step 3. We have

$$S_{\mathcal{B}}(t) f(x) = f(x - t) \exp(K(x - t) - K(x)),$$

with

$$K(x) := \int_0^x k(u) du.$$

We deduce successively

$$S_{\mathcal{B}}(t) \mathcal{A} f(x) = \delta_{x-t=0} \exp(K(x - t) - K(x)) \mathcal{K}[f],$$

next

$$\begin{aligned} g_{t-s}(x) &:= \mathcal{A} S_{\mathcal{B}}(t - s) \mathcal{A} f(x) \\ &= \delta_{x=0} k(t - s) \exp(-K(t - s)) \mathcal{K}[f], \end{aligned}$$

and finally

$$\begin{aligned} (S_{\mathcal{B}}\mathcal{A})^{(*2)}(t) f(x) &= \int_0^t g_{t-s}(x - s) \exp(K(x - s) - K(x)) ds \\ &= \mathbf{1}_{t \geq x} k(t - x) \exp(-K(t - x) - K(x)) \mathcal{K}[f] =: \varphi_t(x) \mathcal{K}[f]. \end{aligned}$$

Summarizing, we have

$$(S_{\mathcal{B}}\mathcal{A})^{(*2)}(t) f = \varphi_t \mathcal{K}[f],$$

with

$$\varphi_t(x) = \psi(t - x) \exp(-K(x)), \quad \psi(u) := \mathbf{1}_{u \geq 0} k(u) e^{-K(u)}.$$

We then compute

$$\hat{\varphi}_z(x) = \hat{\psi}(z) e^{-zx} e^{-K(x)}.$$

On the one hand, using that $K(u) \geq k_0 u - \tilde{k}$, for any $a > a^* = -k_0$ and some $\tilde{k} \in \mathbb{R}$, we have

$$\begin{aligned} \|e^{-zx} e^{-K(x)}\|_{L^1(\nu)} &\leq \int_0^\infty e^{-\Re z x - K(x)} \langle x \rangle^q dx \\ &\leq \int_0^\infty C e^{-(\Re z - a)x} dx \leq \frac{C}{\Re z - a}, \end{aligned}$$

for any $z \in \Delta_a$ and some constant $C \in (0, \infty)$. On the other hand, when furthermore $k' \in L^1(0, \infty)$, we may perform one integration by parts and we get

$$\hat{\psi}(z) = \frac{1}{z} \left(k(0) - \int_0^\infty (k^2(u) - k'(u)) e^{-K(u)} e^{-zu} du \right).$$

As a consequence and similarly as above, we have

$$|\hat{\psi}(z)| \leq \frac{1}{|z|} \left(\|k\|_{L^\infty} + C_1 \|k'\|_{M^1} + C_2 \frac{\|k\|_{L^\infty}^2}{\Re z - a} \right),$$

for any $z \in \Delta_a$, $a > a^*$. By a standard regularization argument, we get the same estimate in the general case when $k' \in M^1([0, \infty))$. All together, we obtain

$$\begin{aligned} \|(R_B(z)\mathcal{A})^2(z)f\|_X &= \|e^{-zx-K(x)}\|_X |\hat{\psi}(z)| |\mathcal{K}[f]| \\ &\leq \frac{C_a}{|z|} \|f\|_X, \end{aligned}$$

for any $z \in \Delta_a$, $a > a^*$ and a constant C_a depending of a , $L^1(\nu)$ and k (through the quantities $\|k\|_{L^\infty}$, $\|k'\|_{M^1}$ and k_1).

Step 4. We observe that

$$\|\hat{\psi}_z\|_Y := \int_0^\infty (|\psi'_z(x)| + |\psi_z(x)|(1+x^{q+1})) dx \leq C,$$

uniformly in $z \in B(0, R)$, from what (iv) immediately follows. \square

Proof of Theorem 4.1. Collecting the information obtained on \mathcal{L} in Corollary 4.4, Lemma 4.5 and using Theorem 3.1, we immediately deduce that (4.4) holds. \square

4.2. Proof of Theorem 1.3 in the case without delay. We present the proof of our main result Theorem 1.3 in the case without delay.

Proof of Theorem 1.3 in the case without delay. We split the proof into three steps.

Step 1. A new formulation. From Lemma 2.3, for a given initial datum $0 \leq f_0 \in X$ with total density of neuron 1, we may write the solution $f \in C([0, \infty); X)$ to the evolution equation (1.1) and the solution F_λ to the stationary problem (1.5) as

$$\begin{aligned} \partial_t f + \partial_x f + k(\lambda\varphi[f])f &= 0, & f(t, 0) &= \varphi[f(t, \cdot)], \\ \partial_x F + k(\lambda M)F &= 0, & F(0) &= M = \varphi[F], \end{aligned}$$

where here and below the λ and x dependency is often removed without any confusion.

Next, we consider the variation function $g := f - F$ which satisfies

$$\partial_t g = -\partial_x g - k(\lambda M)g + (k_\lambda(m) - k_\lambda(M))f$$

complemented with the boundary condition

$$\begin{aligned} g(t, 0) &= \varphi[f(t, \cdot)] - \varphi[F] \\ &= \int_0^\infty k(\lambda\varphi[f])f - \int_0^\infty k(\lambda\varphi[F])F \\ &= \mathcal{M}[g] + (\mathcal{P}_\lambda[f, m] - \mathcal{P}_\lambda[f, M]), \end{aligned}$$

with $\mathcal{M} = \mathcal{M}_\lambda$ defined in (4.2), Considering the boundary condition as a source term again, we deduce that the variation function g satisfies the equation

$$(4.6) \quad \partial_t g = \mathcal{L}g + Z[g],$$

with the nonlinear term $Z[g] := -Q[g] + \delta_0 \mathcal{Q}[g]$ and where

$$\begin{aligned} Q[g] &:= (k_\lambda(\varphi[f]) - k_\lambda(M))f \\ \mathcal{Q}[g] &:= \mathcal{P}_\lambda[f, \varphi[f]] - \mathcal{P}_\lambda[f, M]. \end{aligned}$$

Step 2. The nonlinear term. Using the properties (1.16), the assumption (1.9)-(1.10) and Lemma 2.3, we estimate

$$\begin{aligned} \|Q[g]\|_X &= \|k(\lambda\varphi[f])f - k(\lambda\varphi[F])f\|_{L^1} \\ &\leq \|f\|_{L^\infty} \xi |\varphi[f] - \varphi[F]| \\ &\lesssim \xi \|g\|_{L^1}. \end{aligned}$$

Similarly, for the boundary term, we have

$$|\mathcal{P}_\lambda[f, m] - \mathcal{P}_\lambda[f, M]| \lesssim \xi \|g\|_{L^1}.$$

Step 3. Decay estimate. Thanks to the Duhamel formula, the solution g to the evolution equation (4.6) satisfies

$$g(t) = S_{\mathcal{L}}(t)g_0 + \int_0^t S_{\mathcal{L}}(t-s)Z[g(s)] \, ds.$$

From Theorem 4.1 and the second step, we deduce

$$\begin{aligned} \|g(t)\|_X &\leq C e^{\alpha t} \|g_0\|_X + \int_0^t C e^{\alpha(t-s)} \|Z[g(s)]\|_X \, ds \\ &\lesssim e^{\alpha t} \|g_0\|_X + \xi \int_0^t e^{\alpha(t-s)} \|g(s)\|_X \, ds, \end{aligned}$$

for any $t \geq 0$ and for some constant $\alpha < 0$. Thanks to the Gronwall's lemma, we have

$$\begin{aligned} \|g(t)\|_X &\lesssim e^{\alpha t} \|g_0\|_X + \xi \|g_0\|_X \int_0^t e^{\alpha t} \exp\left\{\int_s^t e^{\alpha(t-r)} \, dr\right\} \, ds \\ &\lesssim e^{\alpha t} \|g_0\|_X + \xi t e^{\alpha t} \|g_0\|_X \\ &\lesssim e^{\alpha' t} \|g_0\|_X, \end{aligned}$$

for some constant $\alpha' \in (\alpha, 0)$. That concludes the proof of Theorem 1.3 in the case without delay. \square

5. CASE WITH DELAY

This section is dedicated to the proof of Theorem 1.3 in the case with delay. Because the arguments are very similar to those of the previous section, they are only briefly explained. Following [13], we get ride of the delay formulation by writing the problem as a system of PDEs. We recall or introduce the notations

$$\mathcal{P}[h, \mu] := \int_0^\infty k_\lambda(\mu) h dx, \quad k_\lambda(\mu) = k(x, \lambda\mu), \quad \mathcal{D}[w] := \int_0^\infty w(y) b(dy).$$

The evolution equation (1.1) writes

$$\partial_t f = -\partial_x f - k_\lambda(m)f + \delta_0 p,$$

with

$$p(t) = \mathcal{P}[f(t), m(t)], \quad m(t) = \int_0^\infty p(t-y) b(dy).$$

Introducing the auxiliary unknown u and the auxiliary equation

$$\partial_t u = -\partial_y u + \delta_0 p,$$

the value of m is given by $m(t) = \mathcal{D}[u(t)]$. All together, the evolution equation rewrites

$$\begin{aligned} \partial_t f &= -\partial_x f - k_\lambda(\mathcal{D}[u])f + \delta_0 p \\ \partial_t u &= -\partial_y u + \delta_0 p, \end{aligned}$$

with $p = \mathcal{P}[f, \mathcal{D}[u]]$.

On the other hand, the stationary equation (1.5) writes

$$0 = -\partial_x F - k_\lambda(M)F + \delta_0 M$$

with $M = \mathcal{P}[F, M]$. Introducing the auxiliary unknown U and equation

$$0 = -\partial_y U + \delta_0 M,$$

the value of M is given by $M = \mathcal{D}[U]$. All together, the stationary equation rewrites

$$\begin{aligned} 0 &= -\partial_x F - k_\lambda(M)F + \delta_0 M \\ 0 &= -\partial_y U + \delta_0 M, \end{aligned}$$

with $M = \mathcal{P}[F, M]$, $M = \mathcal{D}[U]$.

We define now $g := f - F$, $v := u - U$ and we compute

$$\begin{aligned} \partial_t g &= -\partial_x g - k_\lambda(M)g + (k_\lambda(M) - k_\lambda(m))f + \delta_0(p(t) - M) \\ \partial_t v &= -\partial_x v + \delta_0(p(t) - M) \end{aligned}$$

with

$$p(t) - M = \mathcal{P}[g, M] + \int_0^\infty f [k_\lambda(m) - k_\lambda(M)] dx.$$

We first consider the linear system of equations

$$\begin{aligned} \partial_t g &= -\partial_x g - k_\lambda(M)g + \delta_0 \mathcal{P}[g, M] \\ \partial_t v &= -\partial_x v + \delta_0 \mathcal{P}[g, M] \end{aligned}$$

and the associated semigroup S_Λ , where Λ stands for the operator

$$\Lambda(g, v) := (-\partial_x g - k_\lambda(M)g + \delta_0 \mathcal{P}[g, M], -\partial_x v + \delta_0 \mathcal{P}[g, M]).$$

We introduce the space $X := L^1(\nu) \times L^1(\mu)$ with $\mu := e^{-\delta x} dx$ and $\delta > 0$ is defined in condition (1.14).

Lemma 5.1. *The semigroup S_Λ associated to Λ satisfies*

$$S_\Lambda(t) = \mathcal{O}(e^{a't}), \quad a' < 0,$$

in X .

Proof of Lemma 5.1. We write $\Lambda = (\Lambda_1, \Lambda_2)$ and we observe that $\Lambda_1 = \mathcal{L}$, where \mathcal{L} has been defined in the previous section. Because of Theorem 4.1, we already know that

$$\|S_{\Lambda_1}(t)(g_0, v_0)\|_{L^1_q} \lesssim e^{at} \|g_0\|_{L^1_q}, \quad \forall t \geq 0,$$

for any $(g_0, v_0) \in X$, with $\langle g_0 \rangle = 0$. Next, we denote $g(t) := S_{\lambda_\infty}(t)(g_0, v_0)$, $v(t) := S_{\Lambda_2}(t)(g_0, v_0)$, and we compute

$$\begin{aligned} \frac{d}{dt} \int |v(t)| e^{-\delta x} &= -\delta \int |v(t)| e^{-\delta x} + |\mathcal{P}[g(t), M]| \\ &\leq -\delta \int |v(t)| e^{-\delta x} + C e^{at} \|g_0\|_{L^1_q}. \end{aligned}$$

Setting $a' := \max(-\delta, a)$, we conclude by integrating the above differential inequality. \square

The last step consists in proving the nonlinear stability by generalizing again the estimates established in the previous section and in [14, 13]. We write the nonlinear equation as

$$\partial_t(g, v) = \Lambda(g, v) + \mathcal{Z}$$

for a nonlinearity/source term $\mathcal{Z} = (\mathcal{Z}_1, \mathcal{Z}_2)$ with

$$\mathcal{Z}_1 = (k_\lambda(M) - k_\lambda(m))f + \delta_0 \int_0^\infty f [k_\lambda(m) - k_\lambda(M)] dx$$

and

$$\mathcal{Z}_2 = \delta_0 \int_0^\infty f [k_\lambda(m) - k_\lambda(M)] dx.$$

From the assumption (1.9)-(1.10), we have

$$|\mathcal{Z}_2| \leq \delta_0 \|f\|_\infty \int_0^\infty |k(x, \lambda m) - k(x, \lambda M)| dx \leq \delta_0 \xi |m - M|$$

and similarly

$$\|\mathcal{Z}_1\|_{L^1} \leq \xi |m - M|.$$

From the definition of \mathcal{D} , we have

$$|m - M| = |\mathcal{D}[u] - \mathcal{D}[U]| \leq C \|v\|_{L^1(\mu)}.$$

We finally use Duhamel formula

$$(g, v)(t) = S_\Lambda(t)(g_0, v_0) + (S_\Lambda * \mathcal{Z})(t)$$

and then obtain

$$\|(g, v)(t)\| \leq C_0 e^{at} + C_0 \int_0^t e^{a(t-s)} \xi \|(g, v)(s)\| ds.$$

Thanks to the Gronwall lemma, we have

$$\varphi(t) \leq C_0 e^{at} + C_0 \xi \int_0^t e^{a(t-s)} \varphi(s) ds =: \psi(t),$$

which implies

$$\psi'(t) = a\psi(t) + C_0 \xi \varphi(t) \leq (a + C_0 \xi) \psi.$$

We deduce

$$\psi(t) \leq \psi(0) e^{(a+C_0 \xi)t},$$

and then finally obtain

$$\varphi(t) \leq C_0 e^{(a+C_0 \xi)t}.$$

We conclude by taking $\lambda > 0$ small and large enough.

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