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LIMIT LAWS FOR TRANSIENT RANDOM WALKS IN RANDOM ENVIRONMENT ON $\mathbb{Z}$

NATHANAËL ENRIQUEZ, CHRISTOPHE SABOT, AND OLIVIER ZINDY

Abstract. We consider transient random walks in random environment on $\mathbb{Z}$ with zero asymptotic speed. A classical result of Kesten, Kozlov and Spitzer says that the hitting time of the level n converges in law, after a proper normalization, towards a positive stable law, but they do not obtain a description of its parameter. A different proof of this result is presented, that leads to a complete characterization of this stable law. The case of Dirichlet environment turns out to be remarkably explicit.

1. Introduction

One-dimensional random walks in random environment to the nearest neighbors have been introduced in the sixties in order to give a model of DNA replication. In 1975, Solomon gives, in a seminal work [22], a criterion of transience-recurrence for these walks, and shows that three different regimes can be distinguished: the random walk may be recurrent, or transient with a positive asymptotic speed, but it may also be transient with zero asymptotic speed. This last regime, which does not exist among usual random walks, is probably the one which is the less well understood and its study is the purpose of the present paper.

Let us first remind the main existing results concerning the other regimes. In his paper, Solomon computes the asymptotic speed of transient regimes. In 1982, Sinai states, in [20], a limit theorem in the recurrent case. It turns out that the motion in this case is unusually slow since the position of the walk at time n has to be normalized by $(\log n)^2$ in order to present a non trivial limit. In 1986, the limiting law is characterized independently by Kesten [15] and Golosov [10]. Let us notice here that, beyond the interest of his result, Sinai introduces a very powerful and intuitive tool in the study of one-dimensional random walks in random environment. This tool is the potential, which is a function on $\mathbb{Z}$ canonically associated to the random environment. It turns out to be an usual random walk when the transition probabilities at each site are independent and identically distributed (i.i.d.).

Let us now focus on the works about the transient walk with zero asymptotic speed. The main result was obtained by Kesten, Kozlov and Spitzer in [16] who proved that, when normalized by a suitable power of n, the hitting time of the level n converges towards a positive stable law whose index corresponds to the power of n lying in the normalization. Recently, Mayer-Wolf, Roitershtein and Zeitouni [17] generalized this result to the case where the environment is defined by an irreducible Markov chain.

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Our purpose is to characterize the positive stable law in the case of i.i.d. transition probabilities. Let us mention here that the stable limiting law has been characterized in the case of diffusions in random potential when the potential is either a Brownian motion with drift [13], [11] or a Lévy process [21], but we remind here that despite the similarities of both models one cannot transport results from the continuous model to the discrete one.

The proof chooses a radically different approach than previous ones dealing with the transient case. The proofs in [16] and [17] were mainly based on the representation of the trajectory of the walk in terms of branching processes in random environment (with immigration). This encoding was also used by Alili [2] in its study of transient persistent random walks in random environment having zero asymptotic speed. In contrast with these works, our approach relies heavily on Sinai's interpretation of a particle living in a random potential. However, in the recurrent case, the potential one has to deal with is a recurrent random walk and Sinai introduces a notion of valley which does not make sense anymore in our setting where the potential is a (let's say negatively) drifted random walk. Therefore, we introduce a different notion of valley which is closely related to the excursions of this random walk above its past minimum. It turns out that a result of Iglehart [12] provides the asymptotic for the distribution of the tail of the height of these excursions. Now, as soon as one can prove that the hitting time of the level n can be reduced to the time spent by the random walk to cross the high excursions of the potential above its past minimum, between 0 and n, which are well separated in space, an i.i.d. property comes out, and the problem is reduced to the study of the tail of the time spent by the walker to cross a single excursion.

It turns out that the distribution of this tail can be expressed in terms of the expectation of the functional of some meander associated with the random walk defining the potential. Now, this functional is itself related to the constant that appears in Kesten's renewal theorem [14]. These last two facts are contained in [6]. Now, in the case where the transition probabilities follow some Beta distribution a result of Chamayou and Letac [4] gives an explicit formula for this constant which yields finally an explicit formula for the parameter of the positive stable law which is obtained at the limit.

The same technics also allow to derive the convergence of the normalized process to the inverse of a standard stable subordinator. This result can be compared with the scaling limits obtained for the trap model of Bouchaud, see [3] for a review.

Soon after finishing this article, we learnt of an independent work, by Peterson and Zeitouni [18], which, by the study of the fluctuations of the potential, showed that a quenched stable limit law is not possible in the zero asymptotic speed regime.

The paper is organized as follows: the results are stated in Section 2, and the rest of the paper is devoted to the proofs.

2. Notations and main results

Let $\omega := (\omega_i,\, i \in \mathbb{Z})$ be a family of i.i.d. random variables taking values in $(0,1)$ defined on Ω, which stands for the random environment. Denote by P the distribution of ω and by E the corresponding expectation. Conditioning on ω (i.e. choosing an environment), we define the random walk in random environment $(X_n,\, n \geq 0)$ as a nearest-neighbor random walk on $\mathbb{Z}$ with transition probabilities given by ω:

$(X_n,\, n \geq 0)$ is the Markov chain satisfying $X_0 = 0$ and for $n \geq 0$,

$$P_\omega\left(X_{n+1} = x+1 \,|\, X_n = x\right) = \omega_x = 1 - P_\omega\left(X_{n+1} = x-1 \,|\, X_n = x\right).$$

We denote by P_ω the law of $(X_n,\, n \geq 0)$ and E_ω the corresponding expectation. We denote by $\mathbb{P}$ the joint law of $(\omega, (X_n)_{n\geq 0})$. We refer to Zeitouni [23] for an overview of results on random walks in random environment.

In the study of one-dimensional random walks in random environment, an important role is played by the sequence of variables

$$\rho_i := \frac{1-\omega_i}{\omega_i}, \qquad i \in \mathbb{Z}.$$

We now introduce the hitting time $\tau(x)$ of level x for the random walk $(X_n,\, n \geq 0)$,

$$\tau(x) := \inf\{n \geq 1 :\, X_n = x\}, \quad x \in \mathbb{Z}. \tag{2.1}$$

For $\alpha \in (0,1)$, let $\mathcal{S}_\alpha^{ca}$ be a completely asymmetric (actually positive) stable random variable of index α with Laplace transform, for $\lambda > 0$,

$$E[\mathrm{e}^{-\lambda \mathcal{S}_\alpha^{ca}}] = \mathrm{e}^{-\lambda^\alpha}.$$

Moreover, let us introduce the constant C_K describing the tail of Kesten's renewal series, see [14], defined by $R := \sum_{k\geq 0} \rho_0 \ldots \rho_k$:

$$P\{R > x\} \sim \frac{C_K}{x^\kappa}, \qquad x \to \infty. \tag{2.2}$$

Then the main result of the paper can be stated as follows. The symbols "$\xrightarrow{\text{law}}$" denotes the convergence in distribution.

Theorem 1. *Let $\omega := (\omega_i,\, i \in \mathbb{Z})$ be a family of independent and identically distributed random variables such that*

(a) there exists $0 < \kappa < 1$ for which $E\left[\rho_0^\kappa\right] = 1$ and $E\left[\rho_0^\kappa \log^+ \rho_0\right] < \infty$,
(b) the distribution of $\log \rho_0$ is non-lattice.

Then, we have, when n goes to infinity,

$$\frac{\tau(n)}{n^{1/\kappa}} \xrightarrow{\text{law}} 2\left(\frac{\pi\kappa^2}{\sin(\pi\kappa)} C_K^2 E[\rho_0^\kappa \log \rho_0]\right)^{\frac{1}{\kappa}} \mathcal{S}_\kappa^{ca},$$

$$\frac{X_n}{n^\kappa} \xrightarrow{\text{law}} \frac{\sin(\pi\kappa)}{2^\kappa \pi\kappa^2 C_K^2 E[\rho_0^\kappa \log \rho_0]} \left(\frac{1}{\mathcal{S}_\kappa^{ca}}\right)^\kappa.$$

Remark 1. *Note that several probabilistic representations are available to compute C_K numerically, which are equally efficient. The first one was obtained by Goldie [8], a second was conjectured by Siegmund [19], and we obtained a third one in [6], which plays a central role in the proof of the theorem.*

Remark 2. *We think that the method used in this paper could also treat the case $\kappa = 1$ (see Section 9 for conjecture and comments).*

This theorem takes a remarkably explicit form in the case of Dirichlet environment, i.e. when the law of the environment satisfies $\omega_1(\,\mathrm{d}x) = \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}\mathbf{1}_{[0,1]}(x)\,\mathrm{d}x$, with $\alpha, \beta > 0$ and $B(\alpha,\beta) := \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,\mathrm{d}x$, things can be made much more explicit. The assumption of Theorem 1 corresponds to the case where $0 < \alpha - \beta < 1$ and an easy computation leads to $\kappa = \alpha - \beta$.

Thanks to a very nice result of Chamayou and Letac [4] giving the explicit value of C_K in this case, we obtain the following corollary:

Corollary 1. *In the case where* ω_1 *has a distribution* $\mathrm{Beta}(\alpha,\beta)$, *with* $0 < \alpha - \beta < 1$, *Theorem 1 applies with* $\kappa = \alpha - \beta$. *Then, we have, when* n *goes to infinity,*

$$\frac{\tau(n)}{n^{1/\kappa}} \xrightarrow{\text{law}} 2\left(\frac{\pi}{\sin(\pi(\alpha-\beta))}\,\frac{\psi(\alpha)-\psi(\beta)}{B(\alpha,\beta)^2}\right)^{\frac{1}{\alpha-\beta}} \mathcal{S}_\kappa^{ca},$$

$$\frac{X_n}{n^\kappa} \xrightarrow{\text{law}} \frac{\sin(\pi(\alpha-\beta))}{2^{\alpha-\beta}\pi}\,\frac{B(\alpha,\beta)^2}{\psi(\alpha)-\psi(\beta)}\left(\frac{1}{\mathcal{S}_\kappa^{ca}}\right)^\kappa,$$

where ψ *denotes the classical Digamma function,* $\psi(z) := (\log\Gamma)'(z) = \frac{\Gamma'(z)}{\Gamma(z)}$.

Remark 3. *Our technics also allow to derive the convergence of the normalized process. More precisely, under the assumption (a)-(b) of Theorem 1, the law of the process* $\left(n^{-\kappa}X_{\lfloor nt\rfloor},\, t \geq 0\right)$, *defined on the space of càdlàg functions equipped with the uniform topology, converges to the law of*

$$\left(\frac{\sin(\pi\kappa)}{2^\kappa\pi\kappa^2 C_K^2 E\left[\rho_0^\kappa \log\rho_0\right]} Z_t,\, t \geq 0\right),$$

where Z *is the inverse of the* κ*-stable subordinator* Y *satisfying* $\mathbb{E}[\mathrm{e}^{-\lambda Y_t}] = \mathrm{e}^{-t\lambda^\kappa}$, *for all* $\lambda > 0$. *This result can be compared with the scaling limits obtained for the trap model of Bouchaud, see* [3] *for a review.*

In the following, the constant C stands for a positive constant large enough, whose value can change from line to line.

3. Two notions of valleys

Sinai introduced in [20] the notion of valley in a context where the random walk defining the potential was recurrent. We have to do a similar job in our framework where the random walk defining the potential is negatively drifted.

Let us define precisely the potential, denoted by $V = (V(x),\ x \in \mathbb{Z})$. We recall first the following notation

$$\rho_i = \frac{1-\omega_i}{\omega_i}, \qquad i \in \mathbb{Z}.$$

Then, the potential is a function of the environment ω and is defined as follows:

$$V(x) := \begin{cases} \sum_{i=1}^x \log\rho_i & \text{if } x \geq 1, \\ 0 & \text{if } x = 0, \\ -\sum_{i=x+1}^0 \log\rho_i & \text{if } x \leq -1. \end{cases}$$

Furthermore, we consider the weak descending ladder epochs for the potential defined by $e_0 := 0$ and

$$e_i := \inf\{k > e_{i-1} : \ V(k) \le V(e_{i-1})\}, \qquad i \ge 1,$$

which play a crucial role in our proof. Observe that $(e_i - e_{i-1})_{i\ge 1}$ is a family of i.i.d. random variables. Moreover, classical results of fluctuation theory (see [7], p. 396), tell us that, under assumptions (a)-(b) of Theorem 1,

$$E[e_1] < \infty. \tag{3.1}$$

Now, observe that the $((e_i, e_{i+1}])_{i\ge 0}$ stand for the set of excursions of the potential above its past minimum. Let us introduce H_i, the height of the excursion $(e_i, e_{i+1}]$ defined by

$$H_i := \max_{e_i \le k \le e_{i+1}} \left(V(k) - V(e_i)\right),$$

for $i \ge 0$. Note that the $(H_i)_{i\ge 0}$'s are i.i.d. random variables.

The principle of the proof is to notice that the random walk in random environment spends most of its time climbing the high excursions. In order to quantify what "high excursions" are, we need a key result of Iglehart [12] which provides the asymptotic for the distribution of the tail of H_i, namely

$$\forall i \ge 0, \quad P\{H_i > h\} \sim C_I \, \mathrm{e}^{-\kappa h}, \qquad h \to \infty, \tag{3.2}$$

where

$$C_I = \frac{(1 - E[\mathrm{e}^{\kappa V(e_1)}])^2}{\kappa E[\rho_0^\kappa \log \rho_0] E[e_1]}. \tag{3.3}$$

Iglehart's result is actually deduced from a former well-known result of Cramer, whose proof was later simplified by Feller [7], concerning the tail of the maximum $S := \sup\{V(k);\, k \ge 0\}$ which claims that

$$P\{S > h\} \sim C_F \mathrm{e}^{-\kappa h}, \qquad h \to \infty. \tag{3.4}$$

Since S is stochastically bigger than H_0, C_I must be smaller than C_F, and a rather straight argument of Iglehart shows that the ratio between both constants is equal to $1 - E[\mathrm{e}^{\kappa V(e_1)}]$.

Our strategy will be to compute the Laplace transform of the hitting time $\tau(e_n)$ (where $\tau(x)$ is defined by (2.1)) which at the end will be related to $\tau(n)$ by the strong law of large numbers via $E[e_1]$.

Moreover, it appears that the times needed to cross an excursion of height h is roughly of order e^h. Combined with Iglehart's result, it implies that the time to cross an excursion is heavy tailed for $\kappa < 1$. As we know, from classical phenomena arising in the sum of heavy tailed i.i.d. random variables, the particle will spend most of the time at the foot of the very few high excursions, namely those whose height has order $\frac{\log n}{\kappa}$. (Note that, by Iglehart's result, with an overwhelming probability, there are no excursions of height larger than $\frac{(1+\varepsilon)\log n}{\kappa}$, among the n-first excursions.) This explains why the deep valleys we define later are constructed from excursions higher than the critical height $h_n = \frac{(1-\varepsilon)\log n}{\kappa}$. These valleys consist actually in some portion of potential including these excursions. The high excursions are quite seldom and the valleys are likely to be disjoint. In order to deal with almost sure disjoint valleys, we also introduce $*$-valleys which coincide with deep valleys with high probability.

3.1. **The deep valleys.** Let us define the maximal variations of the potential before site x by

$$V^{\uparrow}(x) := \max_{0\le i\le j\le x}(V(j)-V(i)), \qquad x\in\mathbb{N},$$
$$V^{\downarrow}(x) := \min_{0\le i\le j\le x}(V(j)-V(i)), \qquad x\in\mathbb{N}.$$

By extension, we introduce

$$V^{\uparrow}(x,y) := \max_{x\le i\le j\le y}(V(j)-V(i)), \qquad x<y,$$
$$V^{\downarrow}(x,y) := \min_{x\le i\le j\le y}(V(j)-V(i)), \qquad x<y.$$

In order to define deep valleys, we extract from the first n excursions of the potential above its minimum, these whose heights are greater than a critical height h_n, defined by

$$h_n := \frac{(1-\varepsilon)}{\kappa}\log n, \tag{3.5}$$

for some $0<\varepsilon<1/3$, see Figure 1. Let $(\sigma(i))_{i\ge1}$ be the successive indexes of excursions, whose heights are greater than h_n. More precisely,

$$\begin{aligned}\sigma(1) &:= \inf\{i\ge 0 : H_i\ge h_n,\},\\ \sigma(j) &:= \inf\{i>\sigma(j-1) : H_i\ge h_n\}, \qquad j\ge 2,\\ K_n &:= \max\{j\ge 0:\sigma(j)\le n\}.\end{aligned}$$

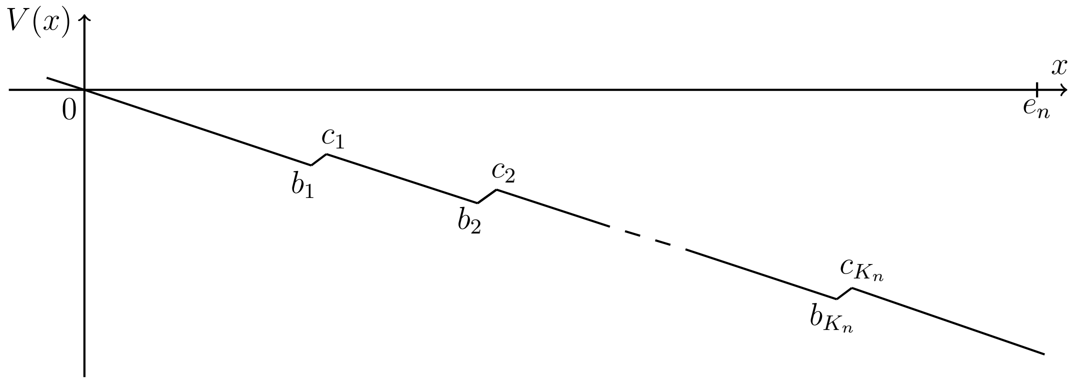

FIGURE 1. Potential and valleys.

We consider now some random variables depending only on n and on the environment, which define the deep valleys.

Definition 1. *For $1\le j\le K_n+1$, let us introduce*

$$\begin{aligned}b_j &:= e_{\sigma(j)},\\ a_j &:= \sup\{k\le b_j : V(k)-V(b_j)\ge D_n\},\\ T_j^{\uparrow} &:= \inf\{k\ge b_j : V(k)-V(b_j)\ge h_n\},\\ \overline{d}_j &:= e_{\sigma(j)+1},\\ c_j &:= \inf\{k\ge b_j : V(k)=\max_{b_j\le x\le \overline{d}_j} V(x)\},\\ d_j &:= \inf\{k\ge \overline{d}_j : V(k)-V(\overline{d}_j)\le -D_n\}.\end{aligned}$$

where $D_n := (1+\frac{1}{\kappa})\log n$. *We call* (a_j, b_j, c_j, d_j) *a deep valley and denote by* $H^{(j)}$ *the height of the* j*-th deep valley.*

Note that all the random variables introduced in this section depend on n, see Figure 2.

Remark 4. *It may happen that two different deep valleys are not disjoint, even if this event is highly improbable as it will be shown in Lemma 4 and Lemma 5 in Subsection 4.1.*

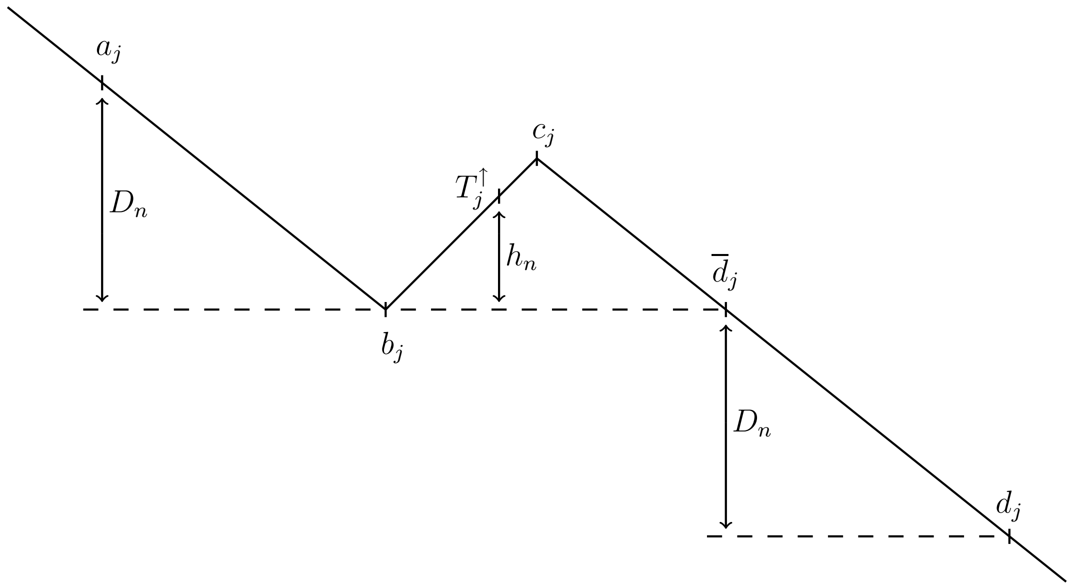

FIGURE 2. Zoom on the j-th valley.

3.2. **The $*$-valleys.** Let us introduce now a subsequence of the deep valleys defined above. It will turn out that both sequences coincide with probability tending to 1 as n goes to infinity. This will be specified in Lemma 6. Let us first introduce

$$\begin{aligned}
\gamma_1^* &:= \inf\{k \geq 0 : V(k) \leq -D_n\},\\
T_1^* &:= \inf\{k \geq \gamma_1^* : V^{\uparrow}(\gamma_1^*, k) \geq h_n\},\\
b_1^* &:= \sup\{k \leq T_1^* : V(k) = \min_{0\leq x\leq T_1^*} V(x)\},\\
a_1^* &:= \sup\{k \leq b_1^* : V(k) - V(b_1^*) \geq D_n\},\\
\overline{d}_1^* &:= \inf\{k \geq T_1^* : V(k) \leq V(b_1^*)\},\\
c_1^* &:= \inf\{k \geq b_1^* : V(k) = \max_{b_1^*\leq x\leq \overline{d}_1^*} V(x)\},\\
d_1^* &:= \inf\{k \geq \overline{d}_1^* : V(k) - V(\overline{d}_1^*) \leq -D_n\}.
\end{aligned}$$

Let us define the following sextuplets of points by iteration

$$(\gamma_j^*, a_j^*, b_j^*, T_j^*, c_j^*, \overline{d}_j^*, d_j^*) := (\gamma_1^*, a_1^*, b_1^*, T_1^*, c_1^*, \overline{d}_1^*, d_1^*) \circ \theta_{d_{j-1}^*}, \qquad j \geq 2,$$

where θ_i denotes the i-shift operator.

Definition 2. *We call a $*$-valley any quadruplet* $(a_j^*, b_j^*, c_j^*, d_j^*)$ *for* $j \geq 1$. *Moreover, we shall denote by* K_n^* *the number of such $*$-valleys before* e_n, *i.e.* $K_n^* := \sup\{j \geq 0 : T_j^* \leq e_n\}$.

It will be made of independent and identically distributed portions of potential (up to some translation).

4. Reduction to a single valley

This section is devoted to the proof of Proposition 1 which tells that the study of $\tau(e_n)$ can be reduced to the analysis of the time spent by the random walk to cross the first deep valley. To ease notations, we introduce $\lambda_n := \frac{\lambda}{n^{1/\kappa}}$.

Proposition 1. *For all n large enough, we have*

$$\mathbb{E}\left[\mathrm{e}^{-\lambda_n\,\tau(e_n)}\right] \;\in\; \left[E\left[E^{b_1}_{\omega,|a_1}\left[\mathrm{e}^{-\lambda_n\tau(d_1)}\right]\right]^{\overline{K}_n}+o(1)\,,\,E\left[E^{b_1}_{\omega,|a_1}\left[\mathrm{e}^{-\lambda_n\tau(d_1)}\right]\right]^{\underline{K}_n}+o(1)\right].$$

where $\underline{K}_n := \lfloor nq_n(1-n^{-\varepsilon/4})\rfloor$, $\overline{K}_n := \lceil nq_n(1+n^{-\varepsilon/4})\rceil$, $q_n := P\{H_0 \geq h_n\}$ and where $E^x_{\omega,|y}$ denotes the quenched law of the random walk in the environment ω, starting at x and reflected at site y.

4.1. **Introducing "good" environments.** Let us define the four following events, that concern exclusively the potential V. The purpose of this subsection is to show that they are realized with an asymptotically overwhelming probability when n goes to infinity. These results will then make it possible to restrict the study of $\tau(e_n)$ to these events.

$$\begin{aligned}
A_1(n) &:= \{e_n < C'n\}\,,\\
A_2(n) &:= \left\{\lfloor nq_n(1-n^{-\varepsilon/4})\rfloor \leq K_n \leq \lceil nq_n(1+n^{-\varepsilon/4})\rceil\right\},\\
A_3(n) &:= \cap_{j=0}^{K_n}\left\{\sigma(j+1)-\sigma(j)\geq n^{1-3\varepsilon}\right\},\\
A_4(n) &:= \cap_{j=1}^{K_n+1}\{d_j-a_j\leq C''\log n\}\,,
\end{aligned}$$

where $\sigma(0) := 0$ (for convenience of notation) and C', C'' stand for positive constants which will be specified below.

In words, $A_1(n)$ allows us to bound the total length of the first n excursions. The event $A_2(n)$ gives a control on the number of deep valleys. The event $A_3(n)$ ensures that the deep valleys are well separated, while $A_4(n)$ bounds finely the length of each of them. Before proving that the A_i's are typical events, let us first give a preliminary result concerning large deviations that we will use throughout the paper.

Lemma 1. *Under assumption (a), large deviations occur for the potential seen as a sum of i.i.d. random variables. Indeed for all $x \geq m := E[\log\rho_0]$ (recall that (a) implies $m<0$) and all $j\geq 1$, we have*

$$P\{V(j)\geq jx\}\leq \exp\{-jI(x)\}, \tag{4.1}$$

with $I(x) := \sup_{t\geq 0}\{tx-\Lambda(t)\}$ and $\Lambda(t) := \log E[\rho_0^t]$. Moreover, the rate function I is lower semicontinuous, satisfies $I(0)>0$ and

$$\inf_{x>0}\frac{I(x)}{x}\geq\kappa. \tag{4.2}$$

Proof. Let us first prove (4.1) which is the upper bound in Cramer's theorem in $\mathbb{R}$, see [5]. Observe first that for all x and every $t \geq 0$, an application of Markov's inequality yields

$$\begin{aligned} P\{V(j) \geq jx\} &= E[\mathbf{1}_{\{V(j)-jx\geq 0\}}] \leq E[\mathrm{e}^{t(V(j)-jx)}] \\ &= \mathrm{e}^{-jtx} E[\mathrm{e}^{t\log\rho_0}]^j = \mathrm{e}^{-j\{tx-\Lambda(t)\}}. \end{aligned} \tag{4.3}$$

Then, we get (4.1) by taking the infimum over $t \geq 0$ in (4.3).

To prove that $I(0) > 0$, observe first that $I(0) = -\inf_{t\geq 0}\Lambda(t)$. Now since the function $g(t) := E[\rho_0^t]$ satisfies $g(0) = g(\kappa) = 1$ (by assumption (a)) and $g'(0) < 0$ (indeed $g'(0) = E[\log\rho_0] < 0$), we get that $\inf_{0\leq t\leq\kappa} g(t) < 1$, which implies $-\inf_{t\geq 0}\Lambda(t) > 0$.

The proof of (4.2) is straightforward. Indeed, recalling that $I(x) = \sup_{t\geq 0}\{tx - \Lambda(t)\}$ for $x > 0$, we have $I(x) \geq \kappa x - \Lambda(\kappa) = \kappa x$, since $\Lambda(\kappa) = 0$. □

Note that the claim of (4.2) appears on page 236 in [23] and that [23] claims an equality under certain assumptions.

Now, let us introduce the following hitting times (for the potential)

$$\begin{aligned} T_h &:= \min\{x \geq 0 :\, V(x) \geq h\}, \qquad h > 0, \\ T_A &:= \min\{x \geq 0 :\, V(x) \in A\}, \qquad A \subset \mathbb{R}. \end{aligned}$$

and prove that the $A_i(n)$'s occur with an overwhelming probability when n tends to infinity.

Lemma 2. *The probability $P\{A_1(n)\}$ converges to 1 when n goes to infinity.*

Proof. It is a direct consequence of the law of large numbers as soon as C' is taken bigger than $E[e_1]$. □

Lemma 3. *The probability $P\{A_2(n)\}$ converges to 1 when n goes to infinity.*

In words, Lemma 3 means that K_n "behaves" like $C_I n^\varepsilon$, when n tends to infinity. In particular, (3.2), which yields $q_n \sim \frac{C_I}{n^{1-\varepsilon}}$, and Lemma 3 imply

$$P\{K_n + 1 \geq 2C_I n^\varepsilon\} \to 0, \qquad n \to \infty. \tag{4.4}$$

Proof. At first, observe that

$$P\Big\{\frac{K_n}{nq_n} \geq 1 + n^{-\varepsilon/4}\Big\} = P\{K_n - nq_n \geq n^{1-\varepsilon/4} q_n\} \leq \frac{\mathrm{Var}(K_n)}{n^{2(1-\varepsilon/4)} q_n^2},$$

the inequality being a consequence of Markov inequality and the fact that K_n follows a binomial distribution of parameter (n, q_n). Moreover, $\mathrm{Var}(K_n) = nq_n(1-q_n) \leq nq_n$ implies

$$P\{\frac{K_n}{nq_n} \geq 1 + n^{-\varepsilon/4}\} \leq \frac{1}{n^{1-\varepsilon/2} q_n}.$$

Now, Iglehart's result (see (3.2)) implies $q_n \sim \frac{C_I}{n^{1-\varepsilon}}$, $n \to \infty$. Therefore we get that $P\{\frac{K_n}{nq_n} \leq 1 + n^{-\varepsilon/4}\}$ converges to 1 when n goes to infinity. Using similar arguments, we get the convergence to 1 of $P\{\frac{K_n}{nq_n} \geq 1 - n^{-\varepsilon/4}\}$. □

Lemma 4. *The probability $P\{A_3(n)\}$ converges to 1 when n goes to infinity.*

Proof. We make first the trivial observation that

$$\begin{aligned} P\{A_3(n)\} &\geq P\{\sigma(j+1)-\sigma(j) \geq n^{1-3\varepsilon},\, 0 \leq j \leq \lfloor 2C_I n^{\varepsilon} \rfloor \,;\, K_n \leq 2C_I n^{\varepsilon}\} \\ &\geq P\{\sigma(j+1)-\sigma(j) \geq n^{1-3\varepsilon},\, 0 \leq j \leq \lfloor 2C_I n^{\varepsilon} \rfloor\} - P\{K_n \geq 2C_I n^{\varepsilon}\}, \end{aligned}$$

the second inequality being a consequence of $P\{A\,;B\} \geq P\{A\} - P\{B^c\}$, for any couple of events A and B. Therefore, recalling (4.4) and using the fact that $(\sigma(j+1)-\sigma(j))_{0\leq j\leq \lfloor 2C_I n^{\varepsilon}\rfloor}$ are i.i.d. random variables, it remains to prove that

$$P\{\sigma(1) \geq n^{1-3\varepsilon}\}^{\lfloor 2C_I n^{\varepsilon}\rfloor} \to 1, \qquad n \to \infty.$$

Since $\sigma(1)$ is a geometrical random variable with parameter q_n, $P\{\sigma(1) \geq n^{1-3\varepsilon}\}$ is equal to $(1-q_n)^{\lceil n^{1-3\varepsilon}\rceil}$, which implies

$$P\{\sigma(1) \geq n^{1-3\varepsilon}\}^{\lfloor 2C_I n^{\varepsilon}\rfloor} = (1-q_n)^{\lfloor 2C_I n^{\varepsilon}\rfloor \lceil n^{1-3\varepsilon}\rceil} \geq \exp\left\{-Cn^{1-2\varepsilon} q_n\right\}.$$

Then, the conclusion follows from (3.2), which implies that $q_n \sim C_I/n^{1-\varepsilon}$, $n \to \infty$. $\square$

Lemma 5. *For C'' large enough, the probability $P\{A_4(n)\}$ converges to 1 when n goes to infinity.*

Proof. Looking at the proof of Lemma 4, we have to prove that $P\{d_j - a_j \geq C'' \log n\}$ is equal to a $o(n^{-\varepsilon})$, $n \to \infty$. Moreover, observing that $d_j - a_j = (d_j - \overline{d}_j) + (\overline{d}_j - T_j^{\uparrow}) + (T_j^{\uparrow} - b_j) + (b_j - a_j)$, the proof of Lemma 5 boils down to showing that, for C'' large enough,

$$P\{d_j - \overline{d}_j \geq \frac{C''}{4}\log n\} = o(n^{-\varepsilon}), \qquad n \to \infty, \tag{4.5}$$

$$P\{\overline{d}_j - T_j^{\uparrow} \geq \frac{C''}{4}\log n\} = o(n^{-\varepsilon}), \qquad n \to \infty, \tag{4.6}$$

$$P\{T_j^{\uparrow} - b_j \geq \frac{C''}{4}\log n\} = o(n^{-\varepsilon}), \qquad n \to \infty, \tag{4.7}$$

$$P\{b_j - a_j \geq \frac{C''}{4}\log n\} = o(n^{-\varepsilon}), \qquad n \to \infty. \tag{4.8}$$

To prove (4.5), we apply the strong Markov property at time $\overline{d}_j$ such that we get $P\{d_j - \overline{d}_j \geq \frac{C''}{4}\log n\} \leq P\{T_{(-\infty,-D_n]} \geq \frac{C''}{4}\log n\}$. Therefore, we have

$$P\{d_j - \overline{d}_j \geq \frac{C''}{4}\log n\} \leq P\{\inf_{0\leq x\leq \frac{C''}{4}\log n} V(x) > -D_n\} \leq P\{V(\frac{C''}{4}\log n) > -D_n\}.$$

Recalling that $D_n := (1+\frac{1}{\kappa})\log n$, we can use Lemma 1, which implies $P\{V(\frac{C''}{4}\log n) > -D_n\} \leq \mathrm{e}^{-\frac{C''}{4}\log n\, I(-\frac{4}{C''}(1+\frac{1}{\kappa}))}$. Then, this inequality implies (4.5) by choosing C'' large enough such that $\frac{C''}{4} I(-\frac{4}{C''}(1+\frac{1}{\kappa})) > \varepsilon$, which is possible since $I(0) > 0$.

To prove (4.6), observe first that (3.2) implies $P\{H^{(j)} > \frac{(1+\varepsilon')}{\kappa}\log n\} \sim n^{-(\varepsilon'+\varepsilon)} = o(n^{-\varepsilon})$, $n \to \infty$. Therefore, we obtain that $P\{\overline{d}_j - T_j^{\uparrow} \geq \frac{C''}{4}\log n\}$ is less or equal than $P\{T_{(-\infty,-\frac{1+\varepsilon'}{\kappa}\log n]} \geq \frac{C''}{4}\log n\} + o(n^{-\varepsilon})$ and conclude the proof with the same arguments we used to treat (4.5).

To get (4.7), observe first that

$$\begin{aligned} P\{T_j^{\uparrow} - b_j \geq \frac{C''}{4}\log n\} &= P\{T_{h_n} \geq \frac{C''}{4}\log n \,|\, H_0 \geq h_n\} \\ &\leq P\{\frac{C''}{4}\log n \leq T_{h_n} < \infty\}/P\{H_0 \geq h_n\}. \end{aligned}$$

Furthermore, Lemma 1 yields

$$\begin{aligned} P\{\frac{C''}{4}\log n \leq T_{h_n} < \infty\} &\leq \sum_{k \geq \frac{C''}{4}\log n} P\{V(k) \geq h_n\} \leq \sum_{k \geq \frac{C''}{4}\log n} \mathrm{e}^{-k\, I(\frac{h_n}{k})} \\ &\leq \sum_{k \geq \frac{C''}{4}\log n} \mathrm{e}^{-k\, I(0)} \leq \frac{C}{n^{\frac{C''}{4} I(0)}}, \end{aligned}$$

the third inequality being a consequence of the fact that the convex rate function $I(\cdot)$ is an increasing function on $(m, +\infty)$. Using (3.2), we get, for all large n,

$$P\{T_j^{\uparrow} - b_j \geq \frac{C''}{4}\log n\} \leq \frac{C}{n^{\frac{C''}{4} I(0) - (1-\varepsilon)}},$$

which yields (4.7), by choosing C'' large enough such that $C'' > \frac{4}{I(0)}$.

For (4.8), observe first that $((V(k-b_j)-V(b_j))_{a_j \leq k \leq b_j}, a_j, b_j)$ has the same distribution as $((V(k))_{a^- \leq k \leq 0}, a^-, 0)$ under $P\{\cdot | V(k) \geq 0, a^- \leq k \leq 0\}$, where $a^- := \sup\{k \leq 0 : V(k) \geq D_n\}$. Then, since $P\{V(k) \geq 0, k \leq 0\} > 0$ and since $(V(-k),\, k \geq 0)$ has the same distribution as $(-V(k),\, k \geq 0)$, we obtain

$$P\{b_j - a_j \geq \frac{C''}{4}\log n\} \leq CP\{T_{(-\infty,-D_n]} > \frac{C''}{4}\log n\} \leq CP\{V(\frac{C''}{4}\log n) > -D_n\}.$$

Now, the arguments are the same as in the proof of (4.5). □

Defining $A(n) := A_1(n) \cap A_2(n) \cap A_3(n) \cap A_4(n)$, a consequence of Lemma 2, Lemma 3, Lemma 4 and Lemma 5, is that

$$P\{A(n)\} \to 1. \tag{4.9}$$

The following lemma tells us that the $*$-valleys coincide with the sequence of deep valleys with an overwhelming probability when n goes to infinity.

Lemma 6. *If $A^*(n) := \{K_n = K_n^*\,;\, (a_j, b_j, c_j, d_j) = (a_j^*, b_j^*, c_j^*, d_j^*),\ 1 \leq j \leq K_n\}$, then we have that the probability $P\{A^*(n)\}$ converges to 1, when n goes to infinity.*

Proof. Since, by definition, the $*$-valleys constitute a subsequence of the deep valleys, Lemma 6 is a consequence of Lemma 4 together with Lemma 5. □

Remark 5. *Another meaning of this result is that, with probability tending to 1, two deep valleys are necessarily disjoint.*

4.2. **Preparatory lemmas.** In this subsection, we develop some technical tools allowing us to improve our understanding of the random walk's behavior. In Lemma 8, we prove that, after exiting a deep valley, the random walk will not come back to another deep valley it has already visited, with probability tending to one. Moreover, Lemma 9 specifies that the random walk typically exits from a $*$-valley on the right, while Lemma 10 shows that the time spent between two deep valleys is negligible. Lemma 11 states that the first valley coincides with the first $*$-valley with probability $1-o(n^{-\varepsilon})$, when n goes to infinity.

4.2.1. *Preliminary estimates for inter-arrival times.* Let us introduce

$$T^{\uparrow}(h) := \min\{x \geq 0 : V^{\uparrow}(x) \geq h\}, \qquad h>0,$$
$$T^{\downarrow}(h) := \min\{x \geq 0 : V^{\downarrow}(x) \leq -h\}, \qquad h>0.$$

Lemma 7. *Under assumptions of Theorem 1, we have, for h large enough,*

$$\mathbb{E}_{|0}\,[\tau_h] \leq C\,\mathrm{e}^{h},$$

where $\mathbb{E}_{|0}$ denotes the expectation under the law $\mathbb{P}_{|0}$ of the random walk in the random environment ω (under P) reflected at 0 *and $\tau_h := \tau(T^{\uparrow}(h)-1)$.*

Proof. Using (Zeitouni [23], formula (2.1.14)), we obtain that $\mathbb{E}_{|0}\,[\tau_h]$ is bounded from above by $E\big[\sum_{0\leq i\leq j<T^{\uparrow}(h)} \mathrm{e}^{V(j)-V(i)}\big]$. Therefore, since $T^{\uparrow}(h) \leq T^{\uparrow}(h)\circ\theta_i$ for any $i\geq 0$ (where θ denotes the shift operator for the environment), we obtain

$$\mathbb{E}_{|0}\,[\tau_h] \leq \sum_{i\geq 0} E\Big[\mathbf{1}_{\{i<T^{\uparrow}(h)\}} \sum_{i\leq j<T^{\uparrow}(h)} \mathrm{e}^{V(j)-V(i)}\Big] \leq \beta_1(h)\,\beta_2(h), \tag{4.10}$$

where

$$\begin{aligned} \beta_1(h) &:= E\big[T^{\uparrow}(h)\big], \\ \beta_2(h) &:= E\Big[\sum_{0\leq j<T^{\uparrow}(h)} \mathrm{e}^{V(j)}\Big]. \end{aligned}$$

To bound $\beta_1(h)$, let us introduce the number N of complete excursions before $T^{\uparrow}(h)$, defined by $N = N(h) := \sup\{i\geq 0 : e_i < T^{\uparrow}(h)\}$. Then, we can write $\beta_1(h) = E[\sum_{i=0}^{N-1}(e_i - e_{i-1}) + (T^{\uparrow}(h) - e_N)]$. Observe that the definition of $T^{\uparrow}(h)$ implies that N is a geometrical random variable with parameter $q = q(h) := P\{H\geq h\}$ and recall that, by (3.2), we have $q \sim C_I\,\mathrm{e}^{-\kappa h}$, $h\to\infty$. Therefore, we get, for h large enough,

$$\begin{aligned} \beta_1(h) &\leq \sum_{k\geq 0}(1-q)^k q\big(kE[e_1|H<h] + E[T_h|H\geq h]\big) \\ &\leq C\sum_{k\geq 0}(1-q)^k q\big(kE[e_1] + E[T_h|H\geq h]\big), \end{aligned}$$

the second inequality being a consequence of the fact that $E[e_1]<\infty$ (see (3.1)) together with $P\{H<h\}\to 1$, $h\to\infty$, by (3.2). By obvious calculations, this yields $\beta_1(h) \leq C(1-q)q^{-1}E[e_1] + E[T_h|H\geq h]$, which implies with (3.2) that

$$\beta_1(h) \leq C\mathrm{e}^{\kappa h} + E[T_h|H\geq h]. \tag{4.11}$$

Now, let us bound $E[T_h|H \geq h]$. For this purpose, we observe first that $E[T_h|H \geq h] \leq C\mathrm{e}^{\kappa h}\sum_{k\geq 0}(k+1)P\{T_h = k+1\,;\,H \geq h\}$. Then, applying the Markov property at time k, we get

$$\begin{aligned} E[T_h|H \geq h] &\leq C\mathrm{e}^{\kappa h}\sum_{k\geq 0}(k+1)E[\mathbf{1}_{\{0<V(k)<h\}}\mathrm{e}^{-\kappa(h-V(k))}] \\ &\leq C\sum_{k\geq 0}(k+1)\sum_{j=0}^{\lfloor h\rfloor}\mathrm{e}^{\kappa(j+1)}P\{V(k)\geq j\}. \end{aligned}$$

By Lemma 1, we have $P\{V(k) \geq j\} \leq \mathrm{e}^{-kI(\frac{j}{k})}$. Now, the fact that $I(\cdot)$ is an increasing function on $\mathbb{R}^+$ along with (4.2) imply

$$P\{V(k)\geq j\} \leq \mathrm{e}^{-\frac{k}{2}I(\frac{j}{k})}\mathrm{e}^{-\frac{k}{2}I(\frac{j}{k})} \leq \mathrm{e}^{-k\frac{I(0)}{2}}\mathrm{e}^{-\kappa\frac{j}{2}}.$$

Since $I(0) > 0$, this yields that there exists $C > 0$ such that, for all large h,

$$E[T_h|H\geq h] \leq C\mathrm{e}^{\frac{\kappa}{2}h}. \tag{4.12}$$

Combining together (4.11) and (4.12), we obtain for h large enough,

$$\beta_1(h) \leq C\mathrm{e}^{\kappa h}. \tag{4.13}$$

Let us now bound $\beta_2(h)$. We introduce first $\mathcal{E}_k := \{\max_{0\leq j\leq k-1} H_j < h\,;\, H_k \geq h\}$ and write

$$\begin{aligned} \beta_2(h) &= \sum_{k\geq 0}E\Big[\mathbf{1}_{\mathcal{E}_k}\sum_{0\leq j<T^{\uparrow}(h)}\mathrm{e}^{V(j)}\Big] \\ &= \sum_{k\geq 0}\Big(\sum_{i=0}^{k-1}E\Big[\mathbf{1}_{\mathcal{E}_k}\mathrm{e}^{V(e_j)}J_i\Big] + E\Big[\mathbf{1}_{\mathcal{E}_k}\mathrm{e}^{V(e_k)}\overline{J}_k\Big]\Big), \end{aligned}$$

where $J_i := \sum_{j=e_i}^{e_{i+1}}\mathrm{e}^{V(j)-V(e_i)}$ for $i \geq 0$ and $\overline{J}_k := \sum_{j=e_k}^{T^{\uparrow}(h)-1}\mathrm{e}^{V(j)-V(e_k)}$ which is well defined on $\mathcal{E}_k$. Observe that $\mathcal{E}_k = \{N(h) = k\}$ and recall that $N(h)$ is a geometrical random variable with parameter $q = q(h) = P\{H \geq h\}$. Then, the Markov property applied at times $(e_j)_{1\leq j\leq k}$ yields that $\beta_2(h)$ is less or equal than

$$\sum_{k\geq 0}(1-q)^k q\Big(E[J_0|H_0<h]\sum_{j=0}^{k-1}E[\mathrm{e}^{V(e_1)}|H_0<h]^j + E[\overline{J}_0|H_0\geq h]E[\mathrm{e}^{V(e_1)}|H_0<h]^k\Big),$$

which implies that $\beta_2(h)$ is bounded from above by

$$\frac{1}{1-E[\mathrm{e}^{V(e_1)}|H_0<h]}E[J_0|H_0<h] + \frac{q}{1-(1-q)E[\mathrm{e}^{V(e_1)}|H_0<h]}E[\overline{J}_0|H_0\geq h].$$

Now, since V is transient to $-\infty$, then H_0 is almost surely finite and $E[\mathrm{e}^{V(e_1)}|H_0 < h] \to E[\mathrm{e}^{V(e_1)}] < 1$, when $h \to \infty$. Recalling that $q = q(h) \to 0$, $h \to \infty$, it follows that

$$\beta_2(h) \leq C\big(E[J_0|H_0<h] + qE[\overline{J}_0|H_0\geq h]\big), \tag{4.14}$$

for h large enough.

Let us first bound $E[\overline{J}_0|H_0 \geq h]$. Recall that if μ denotes the law of ρ_0, thanks to assumption (a) of Theorem 1 we can define the law $\tilde{\mu} = \rho_0^{\kappa}\mu$, and the law $\tilde{P} = \tilde{\mu}^{\otimes\mathbb{Z}}$ which is the law of a sequence of i.i.d. random variables with law $\tilde{\mu}$. The definition of

κ implies that $\int \log \rho \, \tilde{\mu}(d\rho) > 0$. Then, using the explicit form of the Radon-Nykodym derivative between P and $\tilde{P}$, we can write

$$\begin{aligned} E[\overline{J}_0 | H_0 \geq h] &\leq C\mathrm{e}^{\kappa h} \tilde{E}[\mathrm{e}^{-\kappa V(T_h)} \overline{J}_0 \mathbf{1}_{\{H_0 \geq h\}}] \\ &\leq C\tilde{E}\Big[\mathrm{e}^{-\kappa(V(T_h)-h)} \sum_{k=0}^{T_h-1} \mathrm{e}^{V(k)} \mathbf{1}_{\{H_0 \geq h\}}\Big] \\ &\leq C\tilde{E}\Big[\sum_{k=0}^{T_h-1} \mathrm{e}^{V(k)} \mathbf{1}_{\{\min_{0<k<T_h} V(k)>0\}}\Big] \\ &\leq C\tilde{E}\Big[\sum_{k\geq 0} \sum_{p=0}^{\lfloor h \rfloor} \mathrm{e}^{V(k)} \mathbf{1}_{\{p \leq V(k) < p+1\}}\Big] \\ &\leq C \sum_{p=0}^{\lfloor h \rfloor} \mathrm{e}^{p+1} \tilde{E}\Big[\sum_{k\geq 0} \mathbf{1}_{\{p \leq V(k) < p+1\}}\Big]. \end{aligned}$$

Moreover, by Markov property, we have $\tilde{E}[\sum_{k\geq 0} \mathbf{1}_{\{p\leq V(k)<p+1\}}] \leq \tilde{E}[\sum_{k\geq 0} \mathbf{1}_{\{0\leq V(k)<1\}}]$, which is finite since $(V(k))_{k\geq 0}$ has a positive drift under $\tilde{P}$.

Therefore, recalling (4.14) and (3.2), we get

$$\beta_2(h) \leq C(E[J_0 | H_0 < h] + \mathrm{e}^{(1-\kappa)h}) \tag{4.15}$$

and only have to bound $E[J_0 | H_0 < h]$. Recall that $R = \sum_{k\geq 0} \mathrm{e}^{V(k)}$ and observe that $J_0 \leq R$. Moreover, let us denote by $E^{\mathcal{I}}[\cdot]$ the expectation under $P^{\mathcal{I}}\{\cdot\} := P\{\cdot | \mathcal{I}\}$, with $\mathcal{I} := \{H = S\}$. Then, we first observe that $E^{\mathcal{I}}[R | H < h] \geq E[R\, \mathbf{1}_{\{H=S<h\}}] \geq E[J_0 \mathbf{1}_{\{H=S<h\}}]$. Furthermore, since J_0 depends only on $(V(k)\,;\, 0 \leq k \leq e_1)$ and since $P\{V(k) \leq 0\,;\, k \geq 0\} > 0$, we get, by applying the strong Markov property at time e_1, that $E[J_0 \mathbf{1}_{\{H<h\}}] \leq C E^{\mathcal{I}}[R | H < h]$, which implies

$$E[J_0 | H < h] \leq C E^{\mathcal{I}}[R | H < h].$$

Therefore, we only have to prove that $E^{\mathcal{I}}[R | H < h] \leq C\mathrm{e}^{(1-\kappa)h}$. To this aim, we recall first that Corollary 4.1 in [6] implies that, $P^{\mathcal{I}}$-almost surely,

$$E^{\mathcal{I}}[R | \lfloor H \rfloor] \leq C\mathrm{e}^{\lfloor H \rfloor}. \tag{4.16}$$

Now, observe that $E^{\mathcal{I}}[R | H < h] \leq C E^{\mathcal{I}}[R\, \mathbf{1}_{\{H<h\}}]$ and let us write

$$\begin{aligned} E^{\mathcal{I}}[R\, \mathbf{1}_{\{H<h\}}] &\leq \sum_{k=0}^{\lfloor h \rfloor} E^{\mathcal{I}}\Big[\mathbf{1}_{\{\lfloor H \rfloor = k\}} E^{\mathcal{I}}[R | \lfloor H \rfloor = k]\Big] \\ &\leq C \sum_{k=0}^{\lfloor h \rfloor} E^{\mathcal{I}}\Big[\mathbf{1}_{\{\lfloor H \rfloor = k\}} \mathrm{e}^{\lfloor H \rfloor}\Big] \\ &\leq C \sum_{k=0}^{\lfloor h \rfloor} \mathrm{e}^{k} P^{\mathcal{I}}\{\lfloor H \rfloor = k\} \\ &\leq C \sum_{k=0}^{\lfloor h \rfloor} \mathrm{e}^{(1-\kappa)k} \leq C\mathrm{e}^{(1-\kappa)h}, \end{aligned} \tag{4.17}$$

the second inequality is a consequence of (4.16) and the fourth inequality due to the fact that $P^{\mathcal{I}}\{\lfloor H \rfloor = k\} \le c\mathrm{e}^{-\kappa k}$ for some positive constant c. Now assembling (4.10), (4.13), (4.15) and (4.17) concludes the proof of Lemma 7. $\square$

4.2.2. *Important preliminary results.* Before establishing the announced lemmas, we introduce, for any $x, y \in \mathbb{Z}$,

$$\tau(x,y) := \inf\{k \ge 0 : \ X_{\tau(x)+k} = y\}.$$

Recall that $A(n) = A_1(n) \cap A_2(n) \cap A_3(n) \cap A_4(n)$, where the events $(A_i(n))_{1\le i \le 4}$ are defined at the beginning of Subsection 4.1. Then, we have the following results.

Lemma 8. *Defining* $DT(n) := A(n) \cap \bigcap_{j=1}^{K_n} \left\{\tau(d_j, b_{j+1}) < \tau(d_j, \overline{d}_j)\right\}$, *we have*

$$P\{DT(n)\} \to 1, \qquad n \to \infty.$$

Proof. Recalling (4.9), we only have to prove that

$$E\left[\mathbf{1}_{A(n)} \sum_{j=1}^{K_n} P_\omega^{d_j}\{\tau(b_{j+1}) > \tau(\overline{d}_j)\}\right] \to 0. \tag{4.18}$$

By (Zeitouni [23], formula (2.1.4)), we get, for $1 \le j \le K_n$ and for all ω in $A(n)$:

$$P_\omega^{d_j}\left\{\tau(b_{j+1}) > \tau(\overline{d}_j)\right\} = \frac{\sum_{k=d_j}^{b_{j+1}-1} \mathrm{e}^{V(k)}}{\sum_{k=\overline{d}_j}^{b_{j+1}-1} \mathrm{e}^{V(k)}} \le (b_{j+1} - d_j)\mathrm{e}^{V(d_j) - V(\overline{d}_j) + h_n}.$$

Now, let us explain why $b_{K_n+1} - d_{K_n} \le 2n$ with probability tending to 1. Observe first that $b_{K_n+1} - d_{K_n} \le n + T^{\uparrow}(h_n) \circ \theta_n$ if $d_{K_n} \le n$ and $b_{K_n+1} - d_{K_n} \le T^{\uparrow}(h_n) \circ \theta_n$ if $d_{K_n} > n$. Therefore it is sufficient to prove that $P\{T^{\uparrow}(h_n) \ge n\} \to 0$. But using Markov's inequality together with (4.13), we get $P\{T^{\uparrow}(h_n) \ge n\} \le Cn^{-1}\mathrm{e}^{\kappa h_n} \to 0$, when $n \to \infty$.

Moreover we have $b_{j+1} - d_j \le e_n \le C'\, n$ on $A_1(n)$ for $1 \le j \le K_n - 1$ and by definition $V(d_j) - V(\overline{d}_j) \le -D_n$ for $1 \le j \le K_n$. Therefore, we get

$$E\left[\mathbf{1}_{A(n)} \sum_{j=1}^{K_n} P_\omega^{d_j}\{\tau(b_{j+1}) > \tau(\overline{d}_j)\}\right] \le C\, n E[K_n] \mathrm{e}^{-D_n + h_n}.$$

Recalling that $D_n = (1 + \frac{1}{\kappa}) \log n$, $h_n = \frac{1-\varepsilon}{\kappa} \log n$ and since $E[K_n] \le C\, n^{\varepsilon}$ (K_n has a binomial distribution with parameter (n, q_n)), we obtain

$$E\left[\mathbf{1}_{A(n)} \sum_{j=1}^{K_n} P_\omega^{d_j}\{\tau(b_{j+1}) > \tau(\overline{d}_j)\}\right] \quad \le \quad C\, \mathrm{e}^{\varepsilon(1-1/\kappa)\log n},$$

which implies (4.18). $\square$

Lemma 9. *Defining* $DT^*(n) := \bigcap_{j=1}^{K_n^*} \left\{\tau(b_j^*, d_j^*) < \tau(b_j^*, \gamma_j^*)\right\}$, *we have*

$$P\{DT^*(n)\} \to 1, \qquad n \to \infty.$$

Proof. Recall that $A^*(n) = \{K_n = K_n^*\,;\,(a_j,b_j,c_j,d_j) = (a_j^*,b_j^*,c_j^*,d_j^*),\ 1 \le j \le K_n\}$. Then, let us consider $A^\dagger(n) := A^*(n) \cap A_3(n) \cap A_4^*(n)$ to control the $*$-valleys, where $A_4^*(n)$ is defined by $A_4^*(n) := \cap_{j=1}^{K_n^*}\left\{\gamma_{j+1}^* - a_j^* \le C'' \log n\right\} \cap \left\{\gamma_1^* \le C'' \log n\right\}$. Using the same arguments as in the proof of Lemma 5, we can prove that $P\{A_4^*(n)\} \to 1$, $n \to \infty$, for C'' large enough. Then, recalling that Lemma 4 and Lemma 6 imply $P\{A^*(n) \cap A_3(n)\} \to 1$, $n \to \infty$, it remains only to prove that

$$E\left[\mathbf{1}_{A^\dagger(n)} \sum_{j=1}^{K_n} P_\omega^{b_j}\{\tau(d_j) > \tau(\gamma_j^*)\}\right] \to 0. \tag{4.19}$$

Observe that by (Zeitouni [23], formula (2.1.4)) we get, for $1 \le j \le K_n$,

$$\begin{aligned} P_\omega^{b_j}\{\tau(d_j) > \tau(\gamma_j^*)\} &\le (d_j - b_j)\mathrm{e}^{H^{(j)} - (V(\gamma_j^*) - V(b_j))} \\ &\le C \log n\, \mathrm{e}^{H^{(j)} - (V(\gamma_j^*) - V(b_j))}, \end{aligned}$$

the second inequality being a consequence of $\omega \in A^*(n) \cap A_4^*(n)$. Then, to bound $\mathrm{e}^{H^{(j)} - (V(\gamma_j^*) - V(b_j))}$ from above, observe that (3.2) implies $P\{H^{(j)} > \frac{(1+\varepsilon')}{\kappa}\log n\} \sim n^{-(\varepsilon'+\varepsilon)} = o(n^{-\varepsilon})$, $n \to \infty$, for any $\varepsilon' > 0$, which yields that $P\{\bigcap_{j=1}^{K_n}\{H^{(j)} < \frac{(1+\varepsilon')}{\kappa}\log n\}\}$ tends to 1, when n tends to ∞. Therefore, recalling (4.19), we only have to prove that

$$C(\log n) n^{\frac{(1+\varepsilon')}{\kappa}} E\left[\mathbf{1}_{A^\dagger(n)} \sum_{j=1}^{K_n} \mathrm{e}^{-(V(\gamma_j^*) - V(b_j))}\right] \to 0. \tag{4.20}$$

Since $\gamma_j^* - b_{j-1} \le C'' \log n$ on $A_4^*(n)$ and $b_j - b_{j-1} \ge n^{1-3\varepsilon}$ on $A_3(n)$, we get $b_j - \gamma_j^* \ge \frac{1}{2} n^{1-3\varepsilon}$ for $2 \le j \le K_n$ on $A^\dagger(n)$, for all large n. Similarly, $\gamma_0^* \le C'' \log n$ on $A_4^*(n)$ and $b_1 \ge n^{1-3\varepsilon}$ on $A_3(n)$ yield $b_1 - \gamma_1^* \ge \frac{1}{2} n^{1-3\varepsilon}$ on $A^\dagger(n)$. Therefore, recalling the definition of b_j, we can use Lemma 1 and obtain

$$\begin{aligned} P\{A^\dagger(n)\,;\, V(b_j) - V(\gamma_j^*) \ge -n^{\frac{1-3\varepsilon}{2}}\} &\le P\{V(\frac{1}{2} n^{1-3\varepsilon}) \ge -n^{\frac{1-3\varepsilon}{2}}\} \\ &\le \mathrm{e}^{-\frac{n^{1-3\varepsilon}}{2} I\left(-2n^{-\frac{1-3\varepsilon}{2}}\right)} = o(n^{-\varepsilon}), \end{aligned}$$

for any $1 \le j \le K_n$, since $I(0) > 0$. This result implies that the term on the left-hand side in (4.20) is bounded from above by $C \log n\, n^{\frac{(1+\varepsilon')}{\kappa}} E[K_n]\mathrm{e}^{-\frac{n^{1-3\varepsilon}}{2}}$. Then, since $E[K_n] \le C\, n^\varepsilon$, this concludes the proof of Lemma 9. ☐

Lemma 10. *For any* $0 < \eta < \varepsilon(\frac{1}{\kappa} - 1)$, *let us introduce the following event* $IA(n) := A(n) \cap \left\{\sum_{j=1}^{K_n} \tau(d_j, b_{j+1}) < n^{1/\kappa - \eta}\right\}$. *Then, we have*

$$P\{IA(n)\} \to 1, \qquad n \to \infty.$$

Proof. Recalling that $P\{K_n \ge 2C_I n^\varepsilon\} \to 0$, $n \to \infty$, and that Lemma 8 implies that $P\{DT(n)\} \to 1$, $n \to \infty$, it only remains to prove

$$\mathbb{P}\left\{DT(n) \cap \left\{\sum_{j=1}^{\lfloor 2C_I n^\varepsilon \rfloor} \tau(d_j, b_{j+1}) \ge n^{1/\kappa - \eta}\right\}\right\} \to 0, \qquad n \to \infty.$$

Using Markov inequality, we have to prove that

$$\mathbb{E}\Big[\mathbf{1}_{DT(n)}\sum_{j=1}^{\lfloor 2C_I n^{\varepsilon}\rfloor}\tau(d_j,b_{j+1})\Big]=o\left(\frac{1}{n^{1/\kappa-\eta}}\right),\qquad n\to\infty. \tag{4.21}$$

Furthermore, by definition of the event DT (see Lemma 8), we get

$$\begin{aligned}\mathbb{E}\Big[\mathbf{1}_{DT(n)}\sum_{j=1}^{\lfloor 2C_I n^{\varepsilon}\rfloor}\tau(d_j,b_{j+1})\Big] &\le E\Big[\mathbf{1}_{A(n)}\sum_{j=1}^{\lfloor 2C_I n^{\varepsilon}\rfloor}E^{d_j}_{\omega,|\overline{d}_j}[\tau(b_{j+1})]\Big]\\ &\le E\Big[\mathbf{1}_{A(n)}\sum_{j=1}^{\lfloor 2C_I n^{\varepsilon}\rfloor}E^{\overline{d}_j}_{\omega,|\overline{d}_j}[\tau(b_{j+1})]\Big].\end{aligned}$$

Applying successively the strong Markov property at $\overline{d}_{\lfloor 2C_I n^{\varepsilon}\rfloor}$, ..., $\overline{d}_2$, $\overline{d}_1$, this implies

$$\mathbb{E}\Big[\mathbf{1}_{DT(n)}\sum_{j=1}^{\lfloor 2C_I n^{\varepsilon}\rfloor}\tau(d_j,b_{j+1})\Big]\le 2C_I n^{\varepsilon}\mathbb{E}_{|0}[\tau(T^{\uparrow}(h_n)-1)].$$

Therefore, Lemma 7 implies

$$\mathbb{E}\Big[\mathbf{1}_{DT(n)}\sum_{j=1}^{\lfloor 2C_I n^{\varepsilon}\rfloor}\tau(d_j,b_{j+1})\Big]\le Cn^{\varepsilon}\mathrm{e}^{h_n}\le Cn^{\frac{1}{\kappa}-\varepsilon(\frac{1}{\kappa}-1)},$$

which yields (4.21) and concludes the proof, since $0<\eta<\varepsilon(\frac{1}{\kappa}-1)$. □

Lemma 11. *We have*

$$P\{(a_1,b_1,c_1,d_1)\neq(a_1^*,b_1^*,c_1^*,d_1^*)\}=o(n^{-\varepsilon}),\qquad n\to\infty.$$

Proof. Since γ_1^* is a negative record for the potential V, it is sufficient to prove that there is no excursion higher than h_n before γ_1^*. In a first step, we prove that for C large enough

$$P\{\gamma_1^*\ge C\log n\}=o(n^{-\varepsilon}),\qquad n\to\infty. \tag{4.22}$$

Indeed, applying Lemma 1, we get

$$\begin{aligned}P\{\gamma_1^*\ge C\log n\} &\le P\{V(C\log n)\ge -D_n\}\\ &\le \exp\Big\{-CI\big(\frac{1+\kappa^{-1}}{C}\big)\log n\Big\}=o(n^{-\varepsilon}),\end{aligned}$$

by choosing C so that $CI(\frac{1+\kappa^{-1}}{C})>\varepsilon$, which is possible since $I(0)>0$.

In a second step, we prove that the probability that there is an excursion higher than h_n before $C\log n$ is a $o(n^{-\varepsilon})$. Since the number of excursions before $C\log n$ is bounded by $C\log n$, we will prove that

$$P\Big\{\max_{0\le i\le C\log n}H_i\ge h_n\Big\}=o(n^{-\varepsilon}),\qquad n\to\infty. \tag{4.23}$$

But this result is obvious. Indeed, using (3.2) we obtain that the probability term in (4.23) is less than $C\log n\,\mathrm{e}^{-\kappa h_n}=o(n^{-\varepsilon})$. Now assembling (4.22) and (4.23) concludes the proof of Lemma 11.

□

4.3. **Proof of Proposition 1.** Since the time spent on $\mathbb{Z}_-$ is almost surely finite, we reduce our study to the random walk in random environment reflected at 0 and observe that

$$\mathbb{E}\left[\mathrm{e}^{-\lambda_n \tau(e_n)}\right] = \mathbb{E}_{|0}\left[\mathrm{e}^{-\lambda_n \tau(e_n)}\right] + o(1), \qquad n \to \infty,$$

where $\mathbb{E}_{|0}$ denotes the expectation under the law $\mathbb{P}_{|0}$ of the random walk in the random environment ω (under P) reflected at 0.

Furthermore, by definition, $\tau(e_n)$ satisfies

$$\tau(b_1) + \sum_{j=1}^{K_n-1}\{\tau(b_j,d_j)+\tau(d_j,b_{j+1})\} \le \tau(e_n) \le \tau(b_1) + \sum_{j=1}^{K_n}\{\tau(b_j,d_j)+\tau(d_j,b_{j+1})\},$$

such that we easily get that $\mathbb{E}_{|0}\left[\mathrm{e}^{-\lambda_n \tau(e_n)}\right]$ belongs to

$$\left[\mathbb{E}_{|0}\left[\mathrm{e}^{-\lambda_n\left(\tau(b_1)+\sum_{j=1}^{K_n}\{\tau(b_j,d_j)+\tau(d_j,b_{j+1})\}\right)}\right], \mathbb{E}_{|0}\left[\mathrm{e}^{-\lambda_n\left(\tau(b_1)+\sum_{j=1}^{K_n-1}\{\tau(b_j,d_j)+\tau(d_j,b_{j+1})\}\right)}\right]\right].$$

Let us first recall that Lemma 8 and Lemma 10 imply that $P\{DT(n)\cap IA(n)\} \to 1$, $n \to \infty$. Then, we get that the lower bound in the previous interval is equal to

$$\begin{aligned} & \mathbb{E}_{|0}\left[\mathbf{1}_{DT(n)\cap IA(n)}\mathrm{e}^{-\lambda_n\left(\tau(b_1)+\sum_{j=1}^{K_n}\{\tau(b_j,d_j)+\tau(d_j,b_{j+1})\}\right)}\right] + o(1) \\ = \; & \mathbb{E}_{|0}\left[\mathbf{1}_{DT(n)\cap IA(n)}\,\mathrm{e}^{-\lambda_n \sum_{j=1}^{K_n}\tau(b_j,d_j)}\right] + o(1) \\ = \; & \mathbb{E}_{|0}\left[\mathrm{e}^{-\lambda_n \sum_{j=1}^{K_n}\tau(b_j,d_j)}\right] + o(1). \end{aligned}$$

Then, applying the strong Markov property for the random walk successively at $\tau(b_{K_n})$, $\tau(b_{K_n-1})$, $\dots$, $\tau(b_2)$ and $\tau(b_1)$ we get

$$\begin{aligned} \mathbb{E}_{|0}\left[\mathrm{e}^{-\lambda_n \sum_{j=1}^{K_n}\tau(b_j,d_j)}\right] &= E\left[\prod_{j=1}^{K_n} E_{\omega,|0}^{b_j}\left[\mathrm{e}^{-\lambda_n\tau(d_j)}\right]\right] \\ &= E\left[\mathbf{1}_{A^*(n)}\prod_{j=1}^{K_n^*} E_{\omega,|0}^{b_j^*}\left[\mathrm{e}^{-\lambda_n\tau(d_j^*)}\right]\right] + o(1) \\ &= E\left[\prod_{j=1}^{K_n^*} E_{\omega,|0}^{b_j^*}\left[\mathrm{e}^{-\lambda_n\tau(d_j^*)}\right]\right] + o(1), \end{aligned}$$

the second equality being a consequence of Lemma 6. Then, since Lemma 9 implies $\mathbb{P}\{DT^*(n)\} \to 1$, we have

$$\begin{aligned} \mathbb{E}_{|0}\left[\mathrm{e}^{-\lambda_n \sum_{j=1}^{K_n}\tau(b_j,d_j)}\right] &= E\left[\prod_{j=1}^{K_n^*} E_{\omega,|0}^{b_j^*}\left[\mathbf{1}_{DT^*(n)}\,\mathrm{e}^{-\lambda_n\tau(d_j^*)}\right]\right] + o(1) \\ &= E\left[\prod_{j=1}^{K_n^*} E_{\omega,|\gamma_j^*}^{b_j^*}\left[\mathbf{1}_{DT^*(n)}\,\mathrm{e}^{-\lambda_n\tau(d_j^*)}\right]\right] + o(1) \\ &= E\left[\prod_{j=1}^{K_n^*} E_{\omega,|\gamma_j^*}^{b_j^*}\left[\mathrm{e}^{-\lambda_n\tau(d_j^*)}\right]\right] + o(1), \end{aligned}$$

Since $\mathbb{P}\{K_n = K_n^*\} \to 1$, and $\mathbb{P}\{K_n \le \overline{K}_n\} \to 1$, with $\overline{K}_n = \lceil nq_n(1+n^{-\varepsilon/4})\rceil$, we get

$$\mathbb{E}_{|0}\left[\mathrm{e}^{-\lambda_n \tau(e_n)}\right] \ge E\left[\prod_{j=1}^{\overline{K}_n} E_{\omega,|\gamma_j^*}^{b_j^*}\left[\mathrm{e}^{-\lambda_n \tau(d_j^*)}\right]\right] + o(1).$$

Then, applying the strong Markov property (for the potential V) successively at times $\gamma_{\overline{K}_n}^*, \dots, \gamma_2^*$ and observing that the $\left(E_{\omega,|\gamma_j^*}^{b_j^*}\left[\mathrm{e}^{-\lambda_n \tau(d_j^*)}\right]\right)_{1\le j\le \overline{K}_n}$ are i.i.d. random variables, we obtain that

$$\mathbb{E}_{|0}\left[\mathrm{e}^{-\lambda_n \tau(e_n)}\right] \ge E\left[E_{\omega,|\gamma_1^*}^{b_1^*}\mathrm{e}^{-\lambda_n \tau(d_1^*)}\right]^{\overline{K}_n} + o(1).$$

Using Lemma 11 and recalling that $\overline{K}_n = \lceil nq_n(1+n^{-\varepsilon/4})\rceil = O(n^{\varepsilon})$, $n \to \infty$, the strong Markov property applied at γ_1^* yields

$$\mathbb{E}_{|0}\left[\mathrm{e}^{-\lambda_n \tau(e_n)}\right] \ge E\left[E_{\omega,|0}^{b_1}\left[\mathrm{e}^{-\lambda_n \tau(d_1)}\right]\right]^{\overline{K}_n} + o(1).$$

Using similar arguments for the upper bound in the aforementioned interval, we get

$$\mathbb{E}_{|0}\left[\mathrm{e}^{-\lambda_n \tau(e_n)}\right] \in \left[E\left[E_{\omega,|0}^{b_1}\left[\mathrm{e}^{-\lambda_n \tau(d_1)}\right]\right]^{\overline{K}_n} + o(1)\,,\ E\left[E_{\omega,|0}^{b_1}\left[\mathrm{e}^{-\lambda_n \tau(d_1)}\right]\right]^{\underline{K}_n} + o(1)\right].$$

with $\underline{K}_n := \lfloor nq_n(1-n^{-\varepsilon/4})\rfloor$. Furthermore, observe that we have $E\left[E_{\omega,|0}^{b_1}\left[\mathrm{e}^{-\lambda_n \tau(d_1)}\right]\right] = E\left[E_{\omega,|a_1}^{b_1}\left[\mathrm{e}^{-\lambda_n \tau(d_1)}\right]\right] + o(n^{-\varepsilon})$. This is a consequence of Lemma 5, definition of a and the fact that (3.2) implies $P\{H^{(1)} > \frac{(1+\varepsilon')}{\kappa}\log n\} \sim n^{-(\varepsilon'+\varepsilon)} = o(n^{-\varepsilon})$, $n \to \infty$, for any $\varepsilon' > 0$, which gives

$$E\left[P_\omega^{b_1}\{\tau(a_1) < \tau(d_1)\}\right] \le C \log n\, \mathrm{e}^{\frac{(1+\varepsilon')}{\kappa}\log n - D_n} = o(n^{-\varepsilon}).$$

This concludes the proof of Proposition 1. □

5. Annealed Laplace transform for the exit time from a deep valley

This section is devoted to the proof of the linearization. It involves h-processes theory and "sculpture" of a typical deep valley. To ease notations, we shall use a, b, c, and d instead of a_1, b_1, c_1 and d_1. Moreover, let us introduce, for any random variable $Z \ge 0$, the functional

$$R_n(\lambda, Z) := E\left[\frac{1}{1+\frac{\lambda}{n^{1/\kappa}}Z}\right], \tag{5.1}$$

and the two important random variables given by

$$\widehat{M}_1 := \sum_{x=a+1}^{d-1} \mathrm{e}^{-(\widehat{V}(x)-\widehat{V}(b))}, \tag{5.2}$$

$$M_2 := \sum_{x=b}^{d-1} \mathrm{e}^{V(x)-V(c)}, \tag{5.3}$$

where $\widehat{V}$ is defined below in (5.5). Then, the result can be expressed in the following way.

Proposition 2. *For any $\xi > 0$, we have, for all large n,*
$R_n(\mathrm{e}^{\xi}\lambda, 2\mathrm{e}^{H^{(1)}}\widehat{M}_1 M_2)+o(n^{-\varepsilon})\leq E\big[E^b_{\omega,|a}[\mathrm{e}^{-\lambda_n\tau(d)}]\big]\leq R_n(\mathrm{e}^{-\xi}\lambda, 2\mathrm{e}^{H^{(1)}}\widehat{M}_1 M_2)+o(n^{-\varepsilon}).$

5.1. **Two h-processes.** In order to estimate $E^b_{\omega,|a}\left[\mathrm{e}^{-\lambda_n\tau(d)}\right]$, we decompose the passage from b to d into the sum of a random geometrically distributed number, denoted by N, of unsuccessful attempts to reach d from b (i.e. excursions of the particle from b to b which do not hit d), followed by a successful attempt. More precisely, since N is a geometrically distributed random variable with parameter $1-p$ satisfying (see [23], formula (2.1.4))

$$1-p \;=\; \omega_b\,\frac{\mathrm{e}^{V(b)}}{\sum_{x=b}^{d-1}\mathrm{e}^{V(x)}}, \tag{5.4}$$

we can write $\tau(d)=\sum_{i=1}^{N}F_i+G$, where the F_i's are the successive i.i.d. failures and G the first success. The accurate estimation of the time spent by each (successful and unsuccessful) attempt leads us to consider two h-processes where the random walker evolves in two modified potentials, one corresponding to the conditioning on a failure (see the potential $\widehat{V}$ and Lemma 12) and the other to the conditioning on a success (see the potential $\bar{V}$ and Lemma 13).

5.1.1. *The failure case: the h-potential $\widehat{V}$.* Let us fix a realization of ω. To introduce the h-potential $\widehat{V}$, we consider the valley $a<b<c<d$ and define $h(x):=P^x_\omega\{\tau(b)<\tau(d)\}$. For any $b<x<d$, we introduce $\widehat{\omega}_x:=\omega_x\frac{h(x+1)}{h(x)}$. Since h is a harmonic function, we have $1-\widehat{\omega}_x=(1-\omega_x)\frac{h(x-1)}{h(x)}$. Now, $\widehat{V}$ can be defined for $x\geq b$ by

$$\widehat{V}(x):=V(b)+\sum_{i=b+1}^{x}\log\frac{1-\widehat{\omega}_i}{\widehat{\omega}_i}. \tag{5.5}$$

We obtain for any $b\leq x<y<d$,

$$\widehat{V}(y)-\widehat{V}(x)=(V(y)-V(x))+\log\left(\frac{h(x)\,h(x+1)}{h(y)\,h(y+1)}\right). \tag{5.6}$$

Since $h(x)$ is a decreasing function of x by definition, we get

$$\frac{h(x)\,h(x+1)}{h(y)\,h(y+1)}\geq 1. \tag{5.7}$$

Thus we obtain for any $b\leq x<y\leq c$,

$$\widehat{V}(y)-\widehat{V}(x)\geq V(y)-V(x). \tag{5.8}$$

Lemma 12. *For any environment ω, we have*

$$E_\omega\left[F_1\right]=2\,\omega_b\bigg(\sum_{i=a+1}^{b-1}\mathrm{e}^{-(V(i)-V(b))}+\sum_{i=b}^{d-1}\mathrm{e}^{-(\widehat{V}(i)-\widehat{V}(b))}\bigg), \tag{5.9}$$

and

$$E_\omega\left[F_1^2\right]=4\omega_b\,R^{+}+4(1-\omega_b)\,R^{-}, \tag{5.10}$$

where

$$R^+ \;:=\; \sum_{i=b+1}^{d-1}\Big(1+2\sum_{j=b}^{i-2}\mathrm{e}^{\widehat{V}(j)-\widehat{V}(i-1)}\Big)\Big(\mathrm{e}^{-(\widehat{V}(i-1)-\widehat{V}(b))}+2\sum_{j=i+1}^{d-1}\mathrm{e}^{-(\widehat{V}(j-1)-\widehat{V}(b))}\Big),$$
$$R^- \;:=\; \sum_{i=a+1}^{b-1}\Big(1+2\sum_{j=i+2}^{b}\mathrm{e}^{V(j)-V(i+1)}\Big)\Big(\mathrm{e}^{-(V(i+1)-V(b))}+2\sum_{j=a+1}^{i-1}\mathrm{e}^{-(V(j+1)-V(b))}\Big).$$

Remark 6. *Alili* [1] *and Goldsheid* [9] *prove a similar result for a non-conditioned hitting time. Here we give the proof in order to be self-contained.*

Proof. Let us first introduce

$$N_i^+ \;:=\; \sharp\{k<\tau(b): X_k=i-1, X_{k+1}=i\}, \qquad i>b,$$
$$N_i^- \;:=\; \sharp\{k<\tau(b): X_k=i+1, X_{k+1}=i\}, \qquad i<b,$$

and the quenched probability in the environment $\widehat{\omega}$, denoted by $P_{\widehat{\omega}}$. Then, observe that, under $P_{\widehat{\omega}}$, for $i>b$ and conditionally on $N_i^+=x$, N_{i+1}^+ is the sum of x independent geometrical random variables with parameter $\widehat{\omega}_i\in(0,1)$. It means that $E_{\widehat{\omega}}[N_{i+1}^+|N_i^+=x]=\frac{x}{\widehat{\rho}_i}$ and $\mathrm{Var}_{\widehat{\omega}}[N_{i+1}^+|N_i^+=x]=\frac{x}{\widehat{\omega}_i\widehat{\rho}_i^2}$. Similarly, under P_ω, for $i<b$ and conditionally on $N_i^-=x$, N_{i-1}^- is the sum of x independent geometrical random variables with parameter $1-\omega_i$. It means that $E_\omega[N_{i-1}^-|N_i^-=x]=x\rho_i$ and $\mathrm{Var}_\omega[N_{i-1}^-|N_i^-=x]=\frac{x\rho_i^2}{(1-\omega_i)}$.

Since

$$E_\omega[F_1]=2\omega_b\,E_{\widehat{\omega}}[\sum_{b+1}^{d-1}N_i^+]+2(1-\omega_b)\,E_\omega[\sum_{a+1}^{b-1}N_i^-],$$

an easy calculation yields (5.9).

To calculate $E_\omega[F_1^2]$, observe first that

$$E_\omega[F_1^2]=4\omega_b\,E_{\widehat{\omega}}\Big[(\sum_{i=b+1}^{d-1}N_i^+)^2\Big]+4(1-\omega_b)\,E_\omega\Big[(\sum_{i=a+1}^{b-1}N_i^-)^2\Big].$$

Then, it remains to prove that $E_{\widehat{\omega}}[(\sum_{b+1}^{d-1}N_i^+)^2]=R^+$ and $E_\omega[(\sum_{a+1}^{b-1}N_i^-)^2]=R^-$. We will only treat $E_{\widehat{\omega}}[(\sum_{b+1}^{d-1}N_i^+)^2]$, the case of $E_\omega[(\sum_{a+1}^{b-1}N_i^-)^2]$ being similar. We get first

$$E_{\widehat{\omega}}\Big[(\sum_{b+1}^{d-1}N_i^+)^2\Big]=\sum_{i=b+1}^{d-1}E_{\widehat{\omega}}[(N_i^+)^2]+2\sum_{i=b+1}^{d-1}\sum_{j=i+1}^{d-1}E_{\widehat{\omega}}[N_i^+N_j^+]. \tag{5.11}$$

Observe that $E_{\widehat{\omega}}\left[N_i^+N_j^+\right]=E_{\widehat{\omega}}\left[N_i^+E_{\widehat{\omega}}\left[N_j^+\mid N_i^+,\dots,N_{j-1}^+\right]\right]=E_{\widehat{\omega}}\left[N_i^+\frac{N_{j-1}^+}{\widehat{\rho}_{j-1}}\right]$, for $i<j$, so that we get, by iterating,

$$E_{\widehat{\omega}}\left[N_i^+N_j^+\right]=E_{\widehat{\omega}}\left[(N_i^+)^2\right]\frac{1}{\widehat{\rho}_{j-1}\cdots\widehat{\rho}_i}.$$

Recalling (5.11), this yields

$$
\begin{aligned}
E_{\widehat{\omega}}\Big[(\sum_{b+1}^{d-1} N_i^+)^2\Big] &= \sum_{i=b+1}^{d-1} E_{\widehat{\omega}}\left[(N_i^+)^2\right]\left(1+2\sum_{j=i+1}^{d-1}\frac{1}{\widehat{\rho}_i\ldots\widehat{\rho}_{j-1}}\right)\\
&= \sum_{i=b+1}^{d-1} E_{\widehat{\omega}}\left[(N_i^+)^2\right]\left(1+2\sum_{j=i+1}^{d-1}\mathrm{e}^{-(\widehat{V}(j-1)-\widehat{V}(i-1))}\right). \qquad (5.12)
\end{aligned}
$$

Now, observe that $E_{\widehat{\omega}}\left[(N_i^+)^2\right] = E_{\widehat{\omega}}\left[E_{\widehat{\omega}}\left[(N_i^+)^2 | N_{i-1}^+\right]\right]$, which implies

$$
E_{\widehat{\omega}}\left[(N_i^+)^2\right] = E_{\widehat{\omega}}\Big[\sum_{k\geq 1} E_{\widehat{\omega}}[G_1^{(i)}+\cdots+G_k^{(i)}]\mathbf{1}_{\{N_{i-1}^+=k\}}\Big].
$$

Since the $G^{(i)}$'s are i.i.d., we get $E_{\widehat{\omega}}[G_1^{(i)}+\cdots+G_k^{(i)}] = k\mathrm{Var}_{\widehat{\omega}}[G_1^{(i)}]+k^2E_{\widehat{\omega}}[G_1^{(i)}]^2$. Recalling that $E_{\widehat{\omega}}[G_1^{(i)}]=\frac{1}{\widehat{\rho}_{i-1}}$ and $\mathrm{Var}_{\widehat{\omega}}[G_1^{(i)}]=\frac{1}{\widehat{\omega}_{i-1}\widehat{\rho}_{i-1}^2}$, this yields

$$
\begin{aligned}
E_{\widehat{\omega}}\left[(N_i^+)^2\right] &= \frac{E_{\widehat{\omega}}\left[N_{i-1}^+\right]}{\widehat{\omega}_{i-1}\widehat{\rho}_{i-1}^2}+\frac{E_{\widehat{\omega}}\left[(N_{i-1}^+)^2\right]}{\widehat{\rho}_{i-1}^2}\\
&= \frac{1}{\widehat{\omega}_{i-1}\widehat{\rho}_{b+1}\ldots\widehat{\rho}_{i-2}\widehat{\rho}_{i-1}^2}+\frac{E_{\widehat{\omega}}\left[(N_{i-1}^+)^2\right]}{\widehat{\rho}_{i-1}^2}. \qquad (5.13)
\end{aligned}
$$

Denoting $W_{b+1}:=1$ and $W_i:=(\widehat{\rho}_{b+1}\ldots\widehat{\rho}_{i-1})^2E_{\widehat{\omega}}\left[(N_i^+)^2\right]$ for $b+1<i<d$, (5.13) becomes

$$
W_i - W_{i-1} = \frac{\widehat{\rho}_{b+1}\ldots\widehat{\rho}_{i-1}}{\widehat{\omega}_{i-1}} = \widehat{\rho}_{b+1}\ldots\widehat{\rho}_{i-1}+\widehat{\rho}_{b+1}\ldots\widehat{\rho}_{i-2},
$$

the second equality being a consequence of $1/\widehat{\omega}_{i-1}=\widehat{\rho}_{i-1}+1$. Therefore, we have $W_i=\sum_{b+2}^{i}(W_j-W_{j-1})+W_{b+1}=\widehat{\rho}_{b+1}\ldots\widehat{\rho}_{i-1}+2(1+\sum_{b+1}^{i-2}\widehat{\rho}_{b+1}\ldots\widehat{\rho}_j)$, which implies

$$
\begin{aligned}
E_{\widehat{\omega}}\left[(N_i^+)^2\right] &= \frac{1}{\widehat{\rho}_{b+1}\ldots\widehat{\rho}_{i-1}}+2\sum_{j=b}^{i-2}\frac{\widehat{\rho}_{b+1}\ldots\widehat{\rho}_j}{(\widehat{\rho}_{b+1}\ldots\widehat{\rho}_{i-1})^2}\\
&= \mathrm{e}^{-(\widehat{V}(i-1)-\widehat{V}(b))}+2\sum_{j=b}^{i-2}\mathrm{e}^{\widehat{V}(j)-2\widehat{V}(i-1)+\widehat{V}(b)}. \qquad (5.14)
\end{aligned}
$$

Assembling (5.12) and (5.14) yields (5.10). □

5.1.2. *The success case: the h-potential $\bar{V}$.* In a similar way, we introduce the h-potential $\bar{V}$ by considering the valley $a<b<c<d$ and defining $g(x):=P_\omega^x\{\tau(d)<\tau(b)\}$. For any $b<x<d$, we introduce $\bar{\omega}_x:=\omega_x\frac{g(x+1)}{g(x)}$. Since g is a harmonic function, we have $1-\bar{\omega}_x=(1-\omega_x)\frac{g(x-1)}{g(x)}$. Then, $\bar{V}$ can be defined for $x\geq b$ by

$$
\bar{V}(x):=V(b)+\sum_{i=b+1}^{x}\log\frac{1-\bar{\omega}_i}{\bar{\omega}_i}.
$$

We have the following result for any $b<x<y\leq d$,

$$
\bar{V}(y)-\bar{V}(x)=(V(y)-V(x))+\log\left(\frac{g(x)\,g(x+1)}{g(y)g(y+1)}\right). \qquad (5.15)
$$

Since $g(x)$ is a increasing function of x by definition, we get

$$\frac{g(x)\,g(x+1)}{g(y)\,g(y+1)} \le 1. \tag{5.16}$$

Therefore, we obtain for any $c \le x < y \le d$,

$$\bar V(y) - \bar V(x) \le V(y) - V(x). \tag{5.17}$$

Using the same arguments as in the failure case, we get the following result.

Lemma 13. *For any environment* ω, *we have*

$$E_\omega[G] \le 1 + \sum_{i=b+1}^{d} \sum_{j=i}^{d} \mathrm{e}^{\bar V(j) - \bar V(i)}. \tag{5.18}$$

5.2. **Preparatory lemmas.** The study of a typical deep valley involves the following event

$$A_5(n) \;:=\; \left\{\max\{V^{\uparrow}(a,b)\,;\, -V^{\downarrow}(b,c)\,;\, V^{\uparrow}(c,d)\} \le \delta \log n\right\},$$

where $\delta > \varepsilon/\kappa$. In words, $A_5(n)$ ensures that the potential does not have excessive fluctuations in a typical box. Moreover, we have the following result.

Lemma 14. *For any* $\delta > \varepsilon/\kappa$,

$$P\{A_5(n)\} = 1 - o(n^{-\varepsilon}), \qquad n \to \infty.$$

Proof. We easily observe that the proof of Lemma 14 boils down to showing that

$$P\{V^{\uparrow}(a,b) \ge \delta \log n\} \;=\; o(n^{-\varepsilon}), \qquad n \to \infty, \tag{5.19}$$

$$P\{-V^{\downarrow}(b,c) \ge \delta \log n\} \;=\; o(n^{-\varepsilon}), \qquad n \to \infty, \tag{5.20}$$

$$P\{V^{\uparrow}(c,d) \ge \delta \log n\} \;=\; o(n^{-\varepsilon}), \qquad n \to \infty. \tag{5.21}$$

In order to prove (5.21), let us first observe the following trivial inequality

$$P\{V^{\uparrow}(c,d) \ge \delta \log n\} \le P\{V^{\uparrow}(T_1^{\uparrow}, d) \ge \delta \log n\}.$$

Looking at the proof of (4.6), we observe that $P\{d - T_1^{\uparrow} \ge C \log n\} = o(n^{-\varepsilon'})$, for any $\varepsilon' > 0$, by choosing C large enough, depending on ε'. Therefore, we only have to prove that $P\{V^{\uparrow}(T_1^{\uparrow}, T_1^{\uparrow} + C\log n) \ge \delta \log n\} = o(n^{-\varepsilon})$. Then, applying the strong Markov property at time $T_1^{\uparrow}$, we have to prove that $P\{V^{\uparrow}(0, C\log n) \ge \delta \log n\} = o(n^{-\varepsilon})$. Now, by Lemma 1 we get

$$\begin{aligned} P\{V^{\uparrow}(0, C\log n) \ge \delta \log n\} &\le (C\log n)^2 \max_{0 \le k \le C\log n} P\{V(k) \ge \delta \log n\} \\ &\le (C\log n)^2 \max_{0 \le k \le C\log n} \mathrm{e}^{-kI\left(\frac{\delta \log n}{k}\right)} \\ &\le (C\log n)^2 \exp\{-\kappa\delta \log n\}. \end{aligned}$$

Since $\delta > \varepsilon/\kappa$, this yields (5.21).

To get (5.20), observe first that

$$P\{-V^{\downarrow}(b,c) \ge \delta \log n\} \le P\{-V^{\downarrow}(b, T_1^{\uparrow}) \ge \delta \log n\} + P\{-V^{\downarrow}(T_1^{\uparrow}, c) \ge \delta \log n\}.$$

The first term on the right-hand side is equal to $P\{V^{\downarrow}(0,T^{\uparrow}(h_n)) \geq \delta \log n | H_0 > h_n\}$. Recalling that (3.2) implies $P\{H_0 > h_n\} \leq Cn^{-(1-\varepsilon)}$ for all large n and observing the trivial inclusion $\left\{V^{\downarrow}(0,T^{\uparrow}(h_n)) \geq \delta \log n \,;\, H_0 > h_n\right\} \subset \left\{T^{\downarrow}(\delta \log n) < T_{h_n} < T_{(-\infty,0]}\right\}$, it follows that $P\{-V^{\downarrow}(b,T_1^{\uparrow}) \geq \delta \log n\}$ is less or equal than

$$\begin{aligned} & Cn^{1-\varepsilon} P\{T^{\downarrow}(\delta \log n) < T_{h_n} < T_{(-\infty,0]}\} \\ \leq \;& Cn^{1-\varepsilon} \sum_{p=\lfloor \delta \log n \rfloor}^{\lfloor h_n \rfloor} P\{M_\delta \in [p,p+1)\,;\, T^{\downarrow}(\delta \log n) < T_{h_n} < T_{(-\infty,0]}\}, \end{aligned}$$

where $M_\delta := \max\{V(k);\, 0 \leq k \leq T^{\downarrow}(\delta \log n)\}$. Applying the strong Markov property at time $T^{\downarrow}(\delta \log n)$ and recalling (3.4) we bound the term of the previous sum, for $\lfloor \delta \log n \rfloor \leq p \leq \lfloor h_n \rfloor$ and all large n, by

$$P\{S \geq p\}\, P\{S \geq h_n - (p - \delta \log n)\} \leq C \mathrm{e}^{-\kappa p} \mathrm{e}^{-\kappa(h_n - p + \delta \log n))},$$

where $S = \sup\{V(k);\, k \geq 0\}$. Thus, we get $P\{-V^{\downarrow}(b,T_1^{\uparrow}) \geq \delta \log n\} \leq C \lfloor h_n \rfloor n^{-\kappa\delta}$, for all large n, which yields $P\{-V^{\downarrow}(b,T_1^{\uparrow}) \geq \delta \log n\} = o(n^{-\varepsilon})$, $n \to \infty$, since $\delta > \varepsilon/\kappa$. Furthermore, applying the strong Markov property at $T_1^{\uparrow}$, we obtain that $P\{-V^{\downarrow}(T_1^{\uparrow},c) \geq \delta \log n\} \leq P\{-V^{\downarrow}(0,V_{max}) \geq \delta \log n\}$. In a similar way we used before (but easier), we get, by applying the strong Markov property at $T^{\downarrow}(\delta \log n)$, that $P\{-V^{\downarrow}(T_1^{\uparrow},c) \geq \delta \log n\} \leq n^{-\kappa\delta}$ for all large n. Since $\delta > \varepsilon/\kappa$ this yields (5.20).

For (5.19), observe first that $((V(k-b)-V(b))_{a \leq k \leq b}, a, b)$ has the same distribution as $((V(k))_{a^- \leq k \leq 0}, a^-, 0)$ under $P\{\cdot | V(k) \geq 0\,,\, a^- \leq k \leq 0\}$, where $a^- := \sup\{k \leq 0 : V(k) \geq D_n\}$. Then, since $P\{V(k) \geq 0\,,\, k \leq 0\} > 0$ and since $(V(-k)\,,\, k \geq 0)$ has the same distribution as $(-V(k)\,,\, k \geq 0)$, we obtain

$$P\{V^{\uparrow}(a,b) \geq \delta \log n\} \leq CP\{V^{\uparrow}(0,T_{(-\infty,-D_n]}) \geq \delta \log n\}.$$

Now, the arguments are the same as in the proof of (5.21). □

5.3. **Proof of Proposition 2.** Recall that we can write $\tau(d) = \sum_{i=1}^{N} F_i + G$, where the F_i's are the successive i.i.d. failures and G the first success. Then, denoting F_1 by F, we have

$$\begin{aligned} E^b_{\omega,|a}[\mathrm{e}^{-\lambda_n \tau(d)}] &= E^b_{\omega,|a}[\mathrm{e}^{-\lambda_n G}] \sum_{k\geq 0} E^b_{\omega,|a}[\mathrm{e}^{-\lambda_n F}]^k (1-p)p^k \\ &= E^b_{\omega,|a}[\mathrm{e}^{-\lambda_n G}] \frac{1-p}{1-p\,E^b_{\omega,|a}[\mathrm{e}^{-\lambda_n F}]}. \end{aligned} \tag{5.22}$$

In order to replace $E^b_{\omega,|a}[\mathrm{e}^{-\lambda_n F}]$ by $1-\lambda_n E^b_{\omega,|a}[F]$, we observe that $1-\lambda_n E^b_{\omega,|a}[F] \leq E^b_{\omega,|a}[\mathrm{e}^{-\lambda_n F}] \leq 1 - \lambda_n E^b_{\omega,|a}[F] + \frac{\lambda_n^2}{2} E^b_{\omega,|a}[F^2]$, which implies that $E[\frac{1-p}{1-p\,E^b_{\omega,|a}[\mathrm{e}^{-\lambda_n F}]}]$ belongs to

$$\left[E\left[\frac{1-p}{1-p(1-\lambda_n E^b_{\omega,|a}[F])}\right] ; E\left[\frac{1-p}{1-p(1-\lambda_n E^b_{\omega,|a}[F] + \frac{\lambda_n^2}{2} E^b_{\omega,|a}[F^2])}\right]\right].$$

Now, we have to bound $\lambda_n E^b_{\omega,|a}[F^2]$ from above. Then, recalling (5.10), which implies $E^b_{\omega,|a}[F^2] \leq 4(R^+ + R^-)$, we only have to bound R^+ and R^-. By definition of R^+, we obtain

$$R^+ \le (d-b)\left(1+2(d-b)\mathrm{e}^{-\widehat{V}^{\downarrow}(b,d)}\right)\left(3(d-b)\max_{b\le j\le d}\mathrm{e}^{-(\widehat{V}(j)-\widehat{V}(b))}\right). \tag{5.23}$$

Recalling that the estimates (4.5)–(4.8) imply that $P\{d-a\ge C''\log n\}=o(n^{-\varepsilon})$ and that Lemma 14 tells that $P\{A_5(n)\}=1-o(n^{-\varepsilon})$, we are interested in the event $A^{\ddagger}(n):=\{d-a\le C''\log n\}\cap A_5(n)$, whose probability is greater than $1-o(n^{-\varepsilon})$ for n large enough. It allows us to sculpt the deep valley (a,b,c,d), such that we can bound R^+. We are going to show that the fluctuations of $\widehat{V}$ are, in a sense, related to the fluctuations of V controlled by $A_5(n)$. Indeed, (5.8) yields $\widehat{V}^{\downarrow}(b,c)\ge V^{\downarrow}(b,c)\ge -\delta\log n$ on $A^{\ddagger}(n)$. Moreover, (5.6) together with (5.7) imply that $\widehat{V}(y)-\widehat{V}(x)$ is greater than

$$[V(y)-\max_{y\le j\le d-1}V(j)]-[V(x)-\max_{x\le j\le d-1}V(j)]-O(\log_2 n),$$

for any $c\le x\le y\le d$, on $A^{\ddagger}(n)$. Since $V(x)-\max_{x\le j\le d-1}V(j)\le 0$ and $V(y)-\max_{y\le j\le d-1}V(j)\ge -\delta\log n$ on $A^{\ddagger}(n)$, this yields $\widehat{V}^{\downarrow}(c,d)\ge-\delta\log n-O(\log_2 n)$. Furthermore, since (5.6) and (5.7) imply that $\widehat{V}(c)$ is larger than $\max_{b\le j\le c}\widehat{V}(j)-O(\log_2 n)$, assembling $\widehat{V}^{\downarrow}(b,c)\ge-\delta\log n$ with $\widehat{V}^{\downarrow}(c,d)\ge-\delta\log n-O(\log_2 n)$ yield

$$\widehat{V}^{\downarrow}(b,d)\ge-\delta\log n-O(\log_2 n), \tag{5.24}$$

on $A^{\ddagger}(n)$. Therefore, we have, on $A^{\ddagger}(n)$ and for all large n,

$$R^+\le C(\log n)^3 n^{\delta}\max_{b\le j\le d}\mathrm{e}^{-(\widehat{V}(j)-\widehat{V}(b))}. \tag{5.25}$$

Since $\widehat{V}(b)=V(b)$ and (5.7) implies $\widehat{V}(x)\ge V(x)$, for all $b\le x\le c$ (in particular $\widehat{V}(c)\ge V(c)$), it follows from (5.24) that $\widehat{V}(j)-\widehat{V}(b)=(\widehat{V}(j)-\widehat{V}(c))+(\widehat{V}(c)-\widehat{V}(b))\ge h_n-\delta\log n-O(\log_2 n)$, which is greater than 0 for n large enough whenever $\delta<(1-\varepsilon)/\kappa$ (it is possible since $\delta>\varepsilon/\kappa$ and $0<\varepsilon<1/3$). Therefore, recalling (5.25), we obtain, on $A^{\ddagger}(n)$,

$$R^+\le C(\log n)^3 n^{\delta}. \tag{5.26}$$

In a similar way, we prove that $R^-\le C(\log n)^3n^{\delta}$, on $A^{\ddagger}(n)$, which implies that $\lambda_n E^b_{\omega,|a}[F^2]\le C(\log n)^3 n^{\delta-\frac{1}{\kappa}}$. Now, observe that, for any $\xi>0$, $\{\lambda_n E^b_{\omega,|a}[F^2]\le 2(1-\mathrm{e}^{-\xi})\}$ is included in $A^{\ddagger}(n)$, so that $\lambda_n E^b_{\omega,|a}[F^2]\le 2(1-\mathrm{e}^{-\xi})E^b_{\omega,|a}[F]$ with probability larger than $1-o(n^{-\varepsilon})$. Then, introducing

$$R'_n(\lambda):=E\left[\frac{1}{1+\frac{\lambda}{n^{1/\kappa}}\frac{p}{1-p}E^b_{\omega,|a}[F]}\right],$$

we get, for n large enough,

$$R'_n(\lambda)+o(n^{-\varepsilon})\le E\left[\frac{1-p}{1-p\,E^b_{\omega,|a}[\mathrm{e}^{-\lambda_n F}]}\right]\le R'_n(\mathrm{e}^{-\xi}\lambda)+o(n^{-\varepsilon}). \tag{5.27}$$

In order to bound $E^b_{\omega,|a}\left[\mathrm{e}^{-\lambda_n G}\right]$ by below, we observe that $\mathrm{e}^{-x}\ge 1-x$, for any $x\ge 0$, such that $E^b_{\omega,|a}[\mathrm{e}^{-\lambda_n G}]\ge 1-\lambda_n E^b_{\omega,|a}[G]$. Therefore, we only have to bound $E^b_{\omega,|a}[G]$ from above. Recalling (5.18), we get $E^b_{\omega,|a}[G]\le(d-b)^2\mathrm{e}^{\bar{V}^{\uparrow}(b,d)}$. Now, let us bound $\bar{V}^{\uparrow}(b,d)$. We observe first that (5.17) implies $\bar{V}^{\uparrow}(c,d)\le V^{\uparrow}(c,d)$, which yields $\bar V^{\uparrow}(c,d) \le \delta \log n$ on $A^{\ddagger}(n)$. Moreover, (5.15) together with (5.16) imply that $\bar V(y) - \bar V(x)$ is less or equal than

$$[V(y) - \max_{b\le j\le y} V(j)] - [V(x) - \max_{b\le j\le x} V(j)] + O(\log_2 n),$$

for any $b \le x \le y \le c$, on $A^{\ddagger}(n)$. Since $V(y) - \max_{b\le j\le y} V(j) \le 0$ and $V(x) - \max_{b\le j\le x} V(j) \ge -\delta \log n$ on $A^{\ddagger}(n)$, this yields $\bar V^{\uparrow}(b,c) \le \delta \log n + O(\log_2 n)$. Furthermore, (5.17) and the fact that $V(y) \le V(c)$, for $c \le y \le d$, imply that $\bar V(y) \le \bar V(c)$ for $c \le y \le d$. Therefore, we have

$$\bar V^{\uparrow}(b,d) \le \delta \log n + O(\log_2 n),$$

on $A^{\ddagger}(n)$. It means that $E^b_{\omega,|a}[\mathrm{e}^{-\lambda_n G}]$ is greater than $1 - o(n^{-\varepsilon})$ on $A^{\ddagger}(n)$ whenever $\delta < \frac{1}{\kappa} - \varepsilon$, which is possible since $\delta > \varepsilon/\kappa$ and $0 < \varepsilon < 1/3$. Therefore, recalling (5.27), we obtain

$$R_n'(\lambda) + o(n^{-\varepsilon}) \le E\big[E^b_{\omega,|a}[\mathrm{e}^{-\lambda_n \tau(d)}]\big] \le R_n'(\mathrm{e}^{-\xi}\lambda) + o(n^{-\varepsilon}). \tag{5.28}$$

Recalling (5.9) and (5.4), we get

$$R_n(\lambda, 2\widehat{M}_1(\mathrm{e}^{H^{(1)}} M_2 + \omega_b)) \le R_n'(\lambda) \le R_n(\lambda, 2\mathrm{e}^{H^{(1)}} \widehat{M}_1 M_2),$$

where $\widehat{M}_1 := \sum_{x=a+1}^{d-1} \mathrm{e}^{-(\widehat V(x) - \widehat V(b))}$, $M_2 := \sum_{x=b}^{d-1} \mathrm{e}^{V(x)-V(c)}$ and $R_n(\lambda, Z)$ is defined in (5.1). Furthermore, since $\mathrm{e}^{H^{(1)}} \ge n^{\frac{1-\varepsilon}{\kappa}}$, $M_2 \ge 1$ and $\omega_b \le 1$ we obtain that, for any $\xi > 0$ and n large enough, $\omega_b \le (\mathrm{e}^{\xi} - 1)\mathrm{e}^{H^{(1)}} M_2$. Therefore, we have for all large n,

$$R_n(\mathrm{e}^{\xi}\lambda, 2\mathrm{e}^{H^{(1)}} \widehat{M}_1 M_2) \le R_n'(\lambda) \le R_n(\lambda, 2\mathrm{e}^{H^{(1)}} \widehat{M}_1 M_2). \tag{5.29}$$

Now, assembling (5.28) and (5.29) concludes the proof of Proposition 2. □

6. Back to canonical meanders

Recall that $S = \max\{V(k)\,;\, k \ge 0\}$ and let us set $H := \max\{V(k)\,;\, 0 \le k \le T_{\mathbb{R}_-}\} = H_0$, $T_S := \inf\{k \ge 0 : V(k) = S\}$. Moreover, we define $\mathcal{I}_n := \{H = S \ge h_n\} \cap \{V(k) \ge 0\,,\, \forall\, k \le 0\}$, and introduce the random variable $Z := e^S M_1^+ M_2^+$, where $M_1^+ := \sum_{k=a^-}^{T_{h_n/2}} e^{-V(k)}$ and $M_2^+ := \sum_{k=0}^{d^+} e^{V(k)-S}$, with $a^- = \sup\{k \le 0 : V(k) \ge D_n\}$ and $d^+ := \inf\{k \ge e_1 : V(k) - V(e_1) \le -D_n\}$. Then, denoting

$$\mathcal{R}_n(\lambda) := E\Big[\frac{1}{1 + n^{-\frac{1}{\kappa}} 2\lambda Z} | \mathcal{I}_n\Big],$$

we get the following result.

Proposition 3. *For any $\xi > 0$, we have, for n large enough,*

$$\mathcal{R}_n(\mathrm{e}^{\xi}\lambda) + o(n^{-\varepsilon}) \le R_n(\lambda, 2\mathrm{e}^{H^{(1)}} \widehat{M}_1 M_2) \le \mathcal{R}_n(\mathrm{e}^{-\xi}\lambda) + o(n^{-\varepsilon}).$$

Proof. Step 1: we replace $\widehat{M}_1$ by $\widehat{M}_1^T$.

Recall that $A^{\ddagger}(n) = \{d - a \le C'' \log n\} \cap A_5(n)$ and that $P\{A^{\ddagger}(n)\} \ge 1 - o(n^{-\varepsilon})$, for all large n. Now, let us introduce $T(\frac{h_n}{2}) := \inf\{k \ge b : V(k) - V(b) \ge h_n/2\}$ and $\widehat{M}_1^T := \sum_{k=a+1}^{T(\frac{h_n}{2})} e^{-(\widehat V(k) - \widehat V(b))}$. Recalling (5.24), we observe that $\widehat{M}_1 \le \widehat{M}_1^T + C'' \log n \mathrm{e}^{-\frac{h_n}{2} + \delta \log n}$ on $A^{\ddagger}(n)$. This implies that, for any $\xi > 0$, we have $\widehat{M}_1 - \widehat{M}_1^T \le (\mathrm{e}^{\xi} - 1)\widehat{M}_1^T$ for all large n, whenever $\delta < \frac{1-\varepsilon}{2\kappa}$, which is possible since $\delta > \varepsilon/\kappa$ and $0 < \varepsilon < 1/3$. Therefore, we obtain, for n large enough,

$$R_n(\mathrm{e}^{\xi}\lambda, 2\mathrm{e}^{H^{(1)}}\widehat{M}_1^T M_2) + o(n^{-\varepsilon}) \le R_n(\lambda, 2\mathrm{e}^{H^{(1)}}\widehat{M}_1 M_2) \le R_n(\lambda, 2\mathrm{e}^{H^{(1)}}\widehat{M}_1^T M_2).$$

Step 2: we replace $\widehat{M}_1^T$ by M_1^T.

Let us denote $M_1^T := \sum_{k=a+1}^{T(\frac{h_n}{2})} e^{-(V(k)-V(b))}$. Since $T(\frac{h_n}{2}) \le c$, (5.8) implies that $\widehat{M}_1^T \le M_1^T$. Observe that (5.6) with (5.7) imply that $\widehat{V}(y) - \widehat{V}(b) - (V(y) - V(b))$ is less or equal than

$$\log\left(\frac{\sum_{j=b}^{d-1}\mathrm{e}^{V(j)}}{\sum_{j=y}^{d-1}\mathrm{e}^{V(j)}}\frac{\sum_{j=b+1}^{d-1}\mathrm{e}^{V(j)}}{\sum_{j=y+1}^{d-1}\mathrm{e}^{V(j)}}\right) \le \frac{\sum_{j=b}^{y-1}\mathrm{e}^{V(j)}}{\sum_{j=y}^{d-1}\mathrm{e}^{V(j)}} + \frac{\sum_{j=b+1}^{y}\mathrm{e}^{V(j)}}{\sum_{j=y+1}^{d-1}\mathrm{e}^{V(j)}},$$

for any $b \le y \le d$. Therefore, on $A^{\ddagger}(n)$, we obtain $\widehat{V}(y) - \widehat{V}(b) \le (V(y) - V(b)) + C\log n\mathrm{e}^{-\frac{h_n}{2}}$ for any $b \le y \le T(\frac{h_n}{2})$, which yields $\widehat{M}_1^T \ge \exp\{C\log n\,\mathrm{e}^{-\frac{h_n}{2}}\}M_1^T$. Then, for any $\xi > 0$, we obtain that $\widehat{M}_1^T \ge \mathrm{e}^{-\xi}M_1^T$, on $A^{\ddagger}(n)$ and for all large n. This implies

$$R_n(\lambda, 2\mathrm{e}^{H^{(1)}}M_1^T M_2) \le R_n(\lambda, 2\mathrm{e}^{H^{(1)}}\widehat{M}_1^T M_2) \le R_n(\mathrm{e}^{-\xi}\lambda, 2\mathrm{e}^{H^{(1)}}M_1^T M_2) + o(n^{-\varepsilon}).$$

Now, assembling Step 1 and Step 2, we get that, for any $\xi > 0$ and n large enough, $R_n(\lambda, 2\mathrm{e}^{H^{(1)}}\widehat{M}_1 M_2)$ belongs to

$$\left[R_n(\mathrm{e}^{\xi}\lambda, 2\mathrm{e}^{H^{(1)}}M_1^T M_2) + o(n^{-\varepsilon})\,;\, R_n(\mathrm{e}^{-\xi}\lambda, 2\mathrm{e}^{H^{(1)}}M_1^T M_2) + o(n^{-\varepsilon})\right]. \tag{6.1}$$

Step 3: the "good " conditioning.

Let us first observe that $((V(k-b) - V(b))_{a\le k\le d}, a, b, c, d)$ has the same law as $((V(k))_{a^-\le k\le d^+}, a^-, 0, T_H, d^+)$ under $P\{\cdot|\mathcal{I}_n'\}$, where $\mathcal{I}_n' := \{H \ge h_n\,;\, V^{\uparrow}(a^-,0) \le h_n\,;\, V(k) \ge 0\,,\, a^- \le k \le 0\}$. Moreover, we easily obtain that $P\{\{V(k) \ge 0\,,\, a^- \le k \le 0\} \setminus \{V(k) \ge 0\,,\, k \le 0\}\} = O(n^{-(1+\kappa)}) = o(n^{-\varepsilon})$, that $P\{\{H \ge h_n\} \setminus \{H = S\}\} = O(n^{-2(1-\varepsilon)}) = o(n^{-\varepsilon})$ and that $P\{V^{\downarrow}(a^-,0) > h_n\} \le P\{V^{\downarrow}(a^-,0) > \delta\log n\} = o(n^{-\varepsilon})$, with the same arguments as in the proof of Lemma 14. Therefore, we have $P\{\mathcal{I}_n' \triangle \mathcal{I}_n\} = o(n^{-\varepsilon})$. Since $0 \le R_n(\lambda, Y) \le 1$, for any $\lambda > 0$ and any positive random variable Y, this yields

$$R_n(\lambda, 2\mathrm{e}^{H^{(1)}}M_1^T M_2) = \mathcal{R}_n(\lambda) + o(n^{-\varepsilon}). \tag{6.2}$$

Combining (6.1) and (6.2) together concludes the proof of Proposition 3. □

7. Proof of Theorem 1

Observe first that $\mathcal{R}_n(\lambda)$ can be written

$$\mathcal{R}_n(\lambda) = 1 - E\left[1 - \frac{1}{1+2\lambda_n Z}|\mathcal{I}_n\right].$$

Then, we can use Corollary A.1 and Remark A.1 in [6], that together imply

$$E\left[1 - \frac{1}{1+2\lambda_n Z}\,|\,\mathcal{I}_n\right] \sim 2^{\kappa}\frac{\pi\kappa}{\sin(\pi\kappa)}\,\frac{E[M^{\kappa}]^2 C_I}{nP\{H \ge h_n\}}\,\lambda^{\kappa}, \qquad n \to \infty.$$

where the random variable M defined by

$$M := \sum_{k<0} \mathrm{e}^{-V'_k} + \sum_{k\geq 0} \mathrm{e}^{-V''_k}, \tag{7.1}$$

where $(V'_k)_{k<0}$ is distributed as the potential under $P\{\cdot|V_k \geq 0,\, \forall k < 0\}$ while $(V''_k)_{k\geq 0}$ is independent of $(V'_k)_{k<0}$ and is distributed as the potential under $\tilde{P}\{\cdot|V_k > 0,\, \forall k > 0\}$.

Therefore, combining together the results of Proposition 1, Proposition 2, Proposition 3 and recalling that $q_n := P\{H \geq h_n\}$, we get that, for any $\xi > 0$,

$$\begin{aligned}
\liminf_{n\to\infty} \mathbb{E}[\mathrm{e}^{-\lambda_n \tau(e_n)}] &\geq \exp\Big\{ -\Big(2^\kappa \frac{\pi\kappa}{\sin(\pi\kappa)} E[M^\kappa]^2 C_I\Big) (\mathrm{e}^{\xi}\lambda)^\kappa \Big\},\\
\limsup_{n\to\infty} \mathbb{E}[\mathrm{e}^{-\lambda_n \tau(e_n)}] &\leq \exp\Big\{ -\Big(2^\kappa \frac{\pi\kappa}{\sin(\pi\kappa)} E[M^\kappa]^2 C_I\Big) (\mathrm{e}^{-\xi}\lambda)^\kappa \Big\}.
\end{aligned}$$

Since this result holds for any $\xi > 0$, we get,

$$\lim_{n\to\infty} \mathbb{E}[\mathrm{e}^{-\lambda_n \tau(e_n)}] = \exp\Big\{ -\Big(2^\kappa \frac{\pi\kappa}{\sin(\pi\kappa)} E[M^\kappa]^2 C_I\Big) \lambda^\kappa \Big\}.$$

Now, one can be tempted to express the functional $E[M^\kappa]$ in terms of the more usual constant C_K, see (2.2). This is the content of Theorem 2.1 in [6], which yields

$$C_K = E[M^\kappa] C_F = E[M^\kappa] \frac{(1 - E[\mathrm{e}^{\kappa V(e_1)}])}{\kappa E[\rho_0^\kappa \log \rho_0] E[e_1]}.$$

Therefore, the Laplace transform of $n^{-1/\kappa}\tau(e_n)$ is

$$\begin{aligned}
\mathbb{E}[\mathrm{e}^{-\frac{\lambda}{n^{1/\kappa}}\tau(e_n)}] &= \exp\Big\{ -\Big(2^\kappa \frac{\pi\kappa}{\sin(\pi\kappa)} \frac{C_K^2 C_I}{C_F^2}\Big) \lambda^\kappa \Big\} + o(1)\\
&= \exp\Big\{ -\Big(2^\kappa \frac{\pi\kappa^2}{\sin(\pi\kappa)} C_K^2 E[\rho_0^\kappa \log \rho_0] E[e_1]\Big) \lambda^\kappa \Big\} + o(1).
\end{aligned}$$

Finally, since, by the law of large numbers, e_n/n converges almost surely to $E[e_1]$, we conclude that

$$\mathbb{E}[\mathrm{e}^{-\frac{\lambda}{n^{1/\kappa}}\tau(n)}] = \exp\Big\{ -\Big(2^\kappa \frac{\pi\kappa^2}{\sin(\pi\kappa)} C_K^2 E[\rho_0^\kappa \log \rho_0]\Big) \lambda^\kappa \Big\} + o(1).$$

Hence, we obtain that the limit is the positive stable law with index κ and parameter $2^\kappa \frac{\pi\kappa^2}{\sin(\pi\kappa)} C_K^2 E[\rho_0^\kappa \log \rho_0]$. □

We can easily see that we can deduce from this proof the asymptotic of the Laplace transform of the time needed to cross the first $*$-valley.

Corollary 2. *We have*

$$\mathbb{E}\left[1 - e^{-\frac{\lambda}{n^{1/\kappa}}\tau_1^*}\right] \sim 2^\kappa \frac{\pi\kappa}{\sin(\pi\kappa)} \frac{C_U}{nP(H \geq h_n)} \lambda^\kappa,$$

where $C_U = C_I E[M^\kappa]$ is the constant which appears in the tail estimate of Z, in [6].

Remark 7. *This result would hold for a different choice of h_n. Indeed, from the proof of Proposition 2 and Proposition 3 and from Corollary A.1 of* [6]*, we see that the result holds for any choice of h_n such that $e^{h_n} = o(n^{\frac{1}{\kappa}})$ and $h_n \geq n^{\frac{1-\epsilon}{\kappa}}$ for some $0 < \epsilon < 1/3$ (this last condition comes from the technical assumption in (4.1) which is needed in the proof Proposition 2, see (5.26)).*

8. Proof of Corollary 1

We are in the case when the law of the environment satisfies

$$\omega_1(\,\mathrm{d}x) = \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}\mathbf{1}_{[0,1]}(x)\,\mathrm{d}x,$$

with $\alpha, \beta > 0$ and $B(\alpha,\beta) := \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,\mathrm{d}x,$. The assumption of Theorem 1 corresponds to the case where $0 < \alpha - \beta < 1$ and an easy computation leads to $\kappa = \alpha - \beta$. Now, a classical argument of derivation under the sign integral shows that

$$E[\rho_0^\kappa \log \rho_0] = \psi(\alpha) - \psi(\beta),$$

where ψ denotes the classical Digamma function $\psi(z) := (\log \Gamma)'(z) = \frac{\Gamma'(z)}{\Gamma(z)}$. Furthermore, a work of Chamayou and Letac [4] shows that C_K can be made explicit. Indeed, with the notations of [4], ρ_0 follows the law $\beta^{(2)}_{p,q}(dx) := \frac{1}{B(p,q)} x^{p-1}(1+x)^{-p-q}1_{\mathbb{R}_+}(x)dx$ with $p = \beta$ and $q = \alpha$. Then, Example 9 of [4] says that $\sum_{k\geq 1} \mathrm{e}^{V(k)}$ follows the law of $\beta^{(2)}_{\beta,\alpha-\beta}$ having density $\frac{1}{B(\alpha,\beta)} x^{\beta-1}(1+x)^{-\alpha}1_{\mathbb{R}_+}(x)$. But we have $\beta^{(2)}_{\beta,\alpha-\beta}([t,+\infty[) \sim \frac{1}{(\alpha-\beta)B(\alpha,\beta)} \frac{1}{t^{\alpha-\beta}}$, $t \to \infty$. Hence, $C_K = \frac{1}{(\alpha-\beta)B(\alpha,\beta)}$.

9. Toward the case $\kappa = 1$

We intend to treat the critical case $\kappa = 1$ between the transient ballistic and sub-ballistic cases. This case turns out to be more delicate. Indeed, Lemma 7 is replaced by a weaker statement, which says that $\tau(e_n)$ reduces to the time spent by the walker to climb excursions which are higher than $\alpha \log n$ for α arbitrarily small. Due to this reduced height, the new "high" excursions are much more numerous and are not anymore well separated. The definition of the valleys should then be adapted as well as the "linearization" argument, which is more difficult to carry out. Moreover, a result of Goldie [8] gives an explicit formula for the Kesten's renewal constant, namely $C_K = \frac{1}{E[\rho_0 \log \rho_0]}$. As a result, we should obtain, as a consequence of a fluctuation result, the following result, which takes a remarkably simple form: $X_n/(\frac{n}{\log n})$ converges in probability to $E[\rho_0 \log \rho_0]/2$.

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References

[1] Alili, S. (1999). Asymptotic behavior for random walks in random environments. *J. Appl. Probab.* **36**, 334–349.

[2] Alili, S. (1999). Persistent random walks in stationary environment. *J. Stat. Phys.* **94**, 469–494.

[3] Ben Arous, G. and Černý, J. (2006). *Dynamics of trap models*, Ecole d'Éte de Physique des Houches, Session LXXXIII "Mathematical Statistical Physics", pp. 331–394. Elsevier.

[4] Chamayou, J.-F. and Letac G. (1991). Explicit stationary distributions for compositions of random functions and products of random matrices. *J. Theoret. Probab.* **4**, 3–36.

[5] Dembo, A. and Zeitouni, O. (1998). *Large deviations techniques and applications,* (2nd ed.). Applications of Mathematics (New York), 38. Springer-Verlag, New York, 1998.

[6] Enriquez, N., Sabot, C. and Zindy, O. (2009). A probabilistic representation of constants in Kesten's renewal theorem. To appear in *Probability Theory and Related Fields.* `ArXiv math.PR/0703648`.

[7] Feller, W. (1971). *An Introduction to Probability Theory and its Applications,* Vol. II. (2nd ed.). Wiley, New York.

[8] Goldie, C.M. (1991). Implicit renewal theory and tails of solutions of random equations. *Ann. Appl. Proba.* **1**, 126–166.

[9] Goldsheid, I. Ya. (2007). Simple transient random walks in one-dimensional random environment: the central limit theorem. *Probab. Theory Related Fields* **139**, 41–64.

[10] Golosov, A. O. (1986). Limit distributions for random walks in random environments. *Soviet Math. Dokl.* **28**, 18–22.

[11] Hu, Y., Shi, Z. and Yor, M. (2004). Rates of convergence of diffusions with drifted Brownian potentials. *Trans. Amer. Math. Soc.* **351**, 3915–3934.

[12] Iglehart, D.L. (1972). Extreme values in the GI/G/1 queue. *Ann. Math. Statist.* **43**, 627–635.

[13] Kawazu, K. and Tanaka, H. (1997). A diffusion process in a Brownian environment with drift. *J. Math. Soc. Japan* **49**, 189–211.

[14] Kesten, H. (1973). Random difference equations and renewal theory for products of random matrices. *Act. Math.* **131**, 207–248.

[15] Kesten, H. (1986). The limit distribution of Sinai's random walk in random environment. *Physica* **A 138**, 299–309.

[16] Kesten, H., Kozlov, M.V. and Spitzer, F. (1975). A limit law for random walk in a random environment. *Compositio Math.* **30**, 145–168.

[17] Mayer-Wolf, E., Roitershtein, A. and Zeitouni, O. (2004). Limit theorems for one-dimensional transient random walks in Markov environments. *Ann. Inst. Henri Poincaré, Probab. Stat.* **40**, 635–659.

[18] Peterson, J. and Zeitouni, O. (2009). Quenched limits for transient, zero speed one-dimensional random walk in random environment. *Ann. Probab.* **37**, 143–188.

[19] Siegmund, D. (1999). Note on a stochastic recursion. In: State of the art in probability and statistics (Leiden, 1999), 547–554, IMS Lecture Notes Monogr. Ser., 36, Inst. Math. Statist., Beachwood, OH, 2001.

[20] Sinai, Ya.G. (1982). The limiting behavior of a one-dimensional random walk in a random medium. *Th. Probab. Appl.* **27**, 256–268.

[21] Singh, A. (2008). Rates of convergence of a transient diffusion in a spectrally negative Lévy potential. *Ann. Probab.* **36**, 279–318.

[22] Solomon, F. (1975). Random walks in a random environment. *Ann. Probab.* **3**, 1–31.

[23] Zeitouni, O. (2004). *Random Walks in Random Environment*, XXXI summer school in probability, St Flour (2001), *Lecture Notes in Math.* **1837**, pp. 193–312. Springer, Berlin.

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